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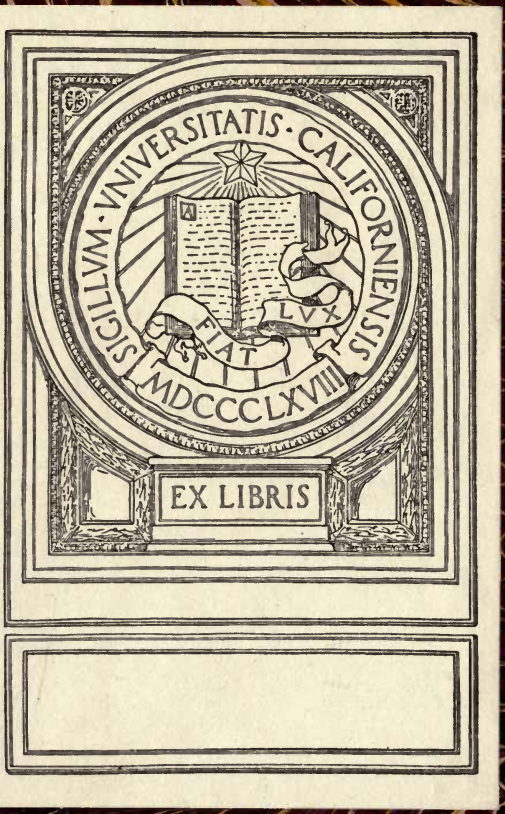
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A TREATISE

ON

CHEMISTRY AND CHEMICAL ANALYSIS

PREPARED FOR STUDENTS OF

THE INTERNATIONAL CORRESPONDENCE SCHOOLS

SCRANTON, PA.

Volume I

ARITHMETIC

ELEMENTARY ALGEBRA AND TRIGONOMETRIC FUNCTIONS

PHYSICS

THEORETICAL CHEMISTRY

WITH PRACTICAL QUESTIONS AND EXAMPLES

First Edition

SCRANTON

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PREFACE.

In the first five of the six volumes composing this set are included all the Instruction and Question Papers used in our Course in Chemistry, including Qualitative and Quantitative Analysis; they give in a clear and concise form not only the theory, but also the practical applications of chemistry and chemical analysis. These volumes constitute a valuable work of reference in connection with the innumerable problems so frequently met with in analytical chemistry and in the manufacturing industries.

The subject of Inorganic Chemistry is very thoroughly covered, and we have introduced a number of simple experiments that will serve to illustrate the text and render the student familiar with the manipulation and use of the apparatus most frequently employed.

That portion of the Course dealing with Organic Chemistry has been so arranged as to render evident the relations existing between the various classes of compounds; for this reason the usual practice of treating the fatty and aromatic derivatives separately has not been adopted, the compounds of both classes being considered together in their appropriate places.

The papers treating on Qualitative and Quantitative Analysis have been prepared with great care; the student who thoroughly studies the text and performs the analyses required will be able to make all the analyses usually performed by analytical chemists.

The method of numbering the pages, cuts, articles, etc. is such that each paper and part is complete in itself; hence, in order to make the indexes intelligible, it was necessary

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to give each paper and part a number. This number is placed at the top of each page, on the headline, opposite the page number; and to distinguish it from the page number, it is preceded by the printer's section mark (§). Consequently, a reference such as page 29, § 18, would be readily found as follows: The back stamp on Vols. I–V shows the sections (i. e., papers) included in the volume, that for Vol. V reading §§ 16–19; hence, look in Vol. V along the headlines until § 18 is found, and then through § 18 until page 29 is found.

The Question Papers are given the same section numbers as the Instruction Papers to which they belong, and are grouped together at the end of the volumes containing the Instruction Papers to which they refer.

The volumes for the present Course, Chemistry, including Qualitative and Quantitative Analysis, are six in number:

Vol. I (§§ 1–5) contains the Instruction and Question Papers on Arithmetic, Elementary Algebra and Trigonometric Functions, Physics, and Theoretical Chemistry.

Vol. II (§§ 6–9) contains the Instruction and Question Papers on Inorganic Chemistry, Parts 1–4.

Vol. III (§§ 10 and 11) contains the Instruction and Question Papers on Qualitative Analysis, Parts 1 and 2.

Vol. IV (§§ 12–15) contains the Instruction and Question Papers on Organic Chemistry, Parts 1–4.

Vol. V (§§ 16–19) contains the Instruction and Question Papers on Quantitative Analysis, Parts 1–4.

Vol. VI contains the answers to the questions and solutions to the examples in the Question Papers. Whenever it has been deemed inadvisable to answer a question, a reference to the proper article in the Instruction Paper has been given, the reading of which will enable the student to answer the question himself. In a number of cases the student is given a solution or some solid substance to analyze. While these analyses constitute a part of the student's work and form a part of the Question Paper, it is not possible to give the answers, owing to the fact that the solutions and solids are varied from time to time.

THE INTERNATIONAL CORRESPONDENCE SCHOOLS.

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ARITHMETIC.

DEFINITIONS.

1. **Arithmetic** is the art of reckoning, or the study of numbers.

2. A **unit** is *one*, or a single thing, as *one* boy, *one* horse, *one* dozen.

3. A **number** is a unit or a collection of units, as *one*, *three* apples, *five* boys.

4. The **unit of a number** is one of the collection of units which constitutes the number. Thus, the unit of *twelve* is *one*, of *twenty* dollars is *one* dollar.

5. A **concrete number** is a number applied to some particular kind of object or quantity, as three *horses*, five *dollars*, ten *pounds*.

6. An **abstract number** is a number that is not applied to any object or quantity, as *three*, *five*, *ten*.

7. **Like numbers** are numbers which express units of the *same kind*, as six *days* and ten *days*, two *feet* and five *feet*.

8. **Unlike numbers** are numbers which express units of *different kinds*, as ten *months* and eight *miles*, seven *dollars* and five *feet*.

NOTATION AND NUMERATION.

9. Numbers are expressed in three ways: (1) by words; (2) by figures; (3) by letters.

10. **Notation** is the art of expressing numbers by figures or letters.

11. **Numeration** is the art of reading the numbers which have been expressed by figures or letters.

12. The **Arabic notation** is the method of expressing numbers by figures. This method employs ten different **figures** to represent numbers, viz.:

Figures	0	1	2	3	4	5	6	7	8	9
Names	*naught, cipher, or zero*	*one*	*two*	*three*	*four*	*five*	*six*	*seven*	*eight*	*nine*

The first character (0) is called **naught, cipher,** or **zero,** and when standing alone has no value.

The other nine figures are called **digits,** and each has a value of its own.

Any whole number is called an **integer.**

13. As there are only ten *figures* used in expressing numbers, each figure must have a different *value* at different times.

14. The value of a figure depends upon its *position* in relation to others.

15. Figures have **simple** values and **local,** or **relative,** values.

16. The **simple** value of a figure is the value it expresses when standing alone.

17. The **local,** or **relative,** value of a figure is the *increased* value it expresses by having other figures placed on its right.

For instance, if we see the figure 6 standing alone, thus 6

we consider it as *six units*, or simply **six.**

Place another 6 to the *left* of it; thus 66

The original figure is still *six units*, but the second figure is *ten times* 6, or 6 **tens.**

If a third 6 be now placed still one place further to the *left*, it is increased in value *ten times* more, thus making it 6 **hundreds** 666

A fourth 6 would be 6 **thousands** 6666

A fifth 6 would be 6 **tens of thousands,** or **sixty thousands** 66666

A sixth 6 would be 6 **hundreds of thousands** . 666666

A seventh 6 would be 6 **millions** 6666666

The entire line of seven figures is read *six millions six hundred sixty-six thousands six hundred sixty-six.*

18. The **increased value** of each of these figures is its *local*, or *relative*, value. Each figure is *ten times* greater in value than the one immediately on its *right.*

19. The **cipher** (0) has no value in itself, but it is useful in determining the place of other figures. To represent the number *four hundred five*, two digits only are necessary, one to represent *four hundred*, and the other to represent *five units;* but if these two digits are placed together, as 45, the 4 (being in the second place) will mean 4 *tens.* To mean 4 *hundreds*, the 4 should have two figures on its right, and a *cipher* is therefore inserted in the place usually given to *tens*, to show that the number is composed of *hundreds* and *units* only, and that there are no *tens. Four hundred five* is therefore expressed as 405. If the number were *four thousand and five*, two ciphers would be inserted; thus, 4005. If it were *four hundred fifty*, it would have the *cipher* at the right-hand side to show that there were no *units*, and only *hundreds and tens;* thus, 450. *Four thousand and fifty* would be expressed 4050, the first cipher indicating that there are no units and the second that there are no hundreds.

20. In *reading* numbers that have been represented by figures, it is usual to *point off* the number into groups of three figures each, beginning with the right-hand, or **units,** column, a comma (,) being used to point off these groups.

Billions.			*Millions.*			*Thousands.*			*Units.*		
Hundreds of Billions.	Tens of Billions.	Billions.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds of Units.	Tens of Units.	Units.
4	3	2 ,	1	9	8 ,	7	6	5 ,	4	3	2

In *pointing off* these figures, begin at the right-hand figure and count—*units*, *tens*, *hundreds;* the next group of three figures is *thousands;* therefore, we insert a comma (,) before beginning with them. Beginning at the figure 5, we say *thousands*, *tens of thousands*, *hundreds of thousands*, and insert another comma. We next read *millions*, *tens of millions*, *hundreds of millions* (insert another comma), *billions*, *tens of billions*, *hundreds of billions.*

The entire line of figures would be read: *four hundred thirty-two billions one hundred ninety-eight millions seven hundred sixty-five thousands four hundred thirty-two.* When we thus *read* a line of figures it is called **numeration,** and if the **numeration** be changed back to *figures*, it is called **notation.**

For instance, the writing of the following figures,

72,584,623,

would be the **notation,** and the **numeration** would be *seventy-two millions five hundred eighty-four thousands six hundred twenty-three.*

21. NOTE.—It is customary to leave the *s* off the words millions, thousands, etc., in cases like the above, both in speaking and writing; hence, the above would usually be expressed seventy-two million five hundred eighty-four thousand six hundred twenty-three.

22. The four fundamental processes of arithmetic are **addition, subtraction, multiplication,** and **division.** They are called fundamental processes because all operations in arithmetic are based upon them.

ADDITION.

23. **Addition** is the *process of finding* the *sum of two or more numbers.* The sign of addition is +. It is read *plus*, and means *more.* Thus, 5 + 6 is read *5 plus 6*, and means that 5 and 6 are to be added.

24. The sign of equality is =. It is read *equals* or *is equal to.* Thus, 5 + 6 = 11 may be read *5 plus 6 equals 11.*

25. *Like* numbers can be added, but *unlike* numbers cannot be added. Thus, 6 dollars can be added to 7 dollars, and the sum will be 13 dollars; but 6 dollars cannot be added to 7 feet.

26. The following table gives the sum of any two numbers from 1 to 12:

1 and 1 is 2	2 and 1 is 3	3 and 1 is 4	4 and 1 is 5
1 and 2 is 3	2 and 2 is 4	3 and 2 is 5	4 and 2 is 6
1 and 3 is 4	2 and 3 is 5	3 and 3 is 6	4 and 3 is 7
1 and 4 is 5	2 and 4 is 6	3 and 4 is 7	4 and 4 is 8
1 and 5 is 6	2 and 5 is 7	3 and 5 is 8	4 and 5 is 9
1 and 6 is 7	2 and 6 is 8	3 and 6 is 9	4 and 6 is 10
1 and 7 is 8	2 and 7 is 9	3 and 7 is 10	4 and 7 is 11
1 and 8 is 9	2 and 8 is 10	3 and 8 is 11	4 and 8 is 12
1 and 9 is 10	2 and 9 is 11	3 and 9 is 12	4 and 9 is 13
1 and 10 is 11	2 and 10 is 12	3 and 10 is 13	4 and 10 is 14
1 and 11 is 12	2 and 11 is 13	3 and 11 is 14	4 and 11 is 15
1 and 12 is 13	2 and 12 is 14	3 and 12 is 15	4 and 12 is 16
5 and 1 is 6	6 and 1 is 7	7 and 1 is 8	8 and 1 is 9
5 and 2 is 7	6 and 2 is 8	7 and 2 is 9	8 and 2 is 10
5 and 3 is 8	6 and 3 is 9	7 and 3 is 10	8 and 3 is 11
5 and 4 is 9	6 and 4 is 10	7 and 4 is 11	8 and 4 is 12
5 and 5 is 10	6 and 5 is 11	7 and 5 is 12	8 and 5 is 13
5 and 6 is 11	6 and 6 is 12	7 and 6 is 13	8 and 6 is 14
5 and 7 is 12	6 and 7 is 13	7 and 7 is 14	8 and 7 is 15
5 and 8 is 13	6 and 8 is 14	7 and 8 is 15	8 and 8 is 16
5 and 9 is 14	6 and 9 is 15	7 and 9 is 16	8 and 9 is 17
5 and 10 is 15	6 and 10 is 16	7 and 10 is 17	8 and 10 is 18
5 and 11 is 16	6 and 11 is 17	7 and 11 is 18	8 and 11 is 19
5 and 12 is 17	6 and 12 is 18	7 and 12 is 19	8 and 12 is 20
9 and 1 is 10	10 and 1 is 11	11 and 1 is 12	12 and 1 is 13
9 and 2 is 11	10 and 2 is 12	11 and 2 is 13	12 and 2 is 14
9 and 3 is 12	10 and 3 is 13	11 and 3 is 14	12 and 3 is 15
9 and 4 is 13	10 and 4 is 14	11 and 4 is 15	12 and 4 is 16
9 and 5 is 14	10 and 5 is 15	11 and 5 is 16	12 and 5 is 17
9 and 6 is 15	10 and 6 is 16	11 and 6 is 17	12 and 6 is 18
9 and 7 is 16	10 and 7 is 17	11 and 7 is 18	12 and 7 is 19
9 and 8 is 17	10 and 8 is 18	11 and 8 is 19	12 and 8 is 20
9 and 9 is 18	10 and 9 is 19	11 and 9 is 20	12 and 9 is 21
9 and 10 is 19	10 and 10 is 20	11 and 10 is 21	12 and 10 is 22
9 and 11 is 20	10 and 11 is 21	11 and 11 is 22	12 and 11 is 23
9 and 12 is 21	10 and 12 is 22	11 and 12 is 23	12 and 12 is 24

This table should be carefully committed to memory. Since 0 has no value, the sum of any number and 0 is the number itself; thus 17 and 0 is 17.

27. For addition, place the numbers to be added directly under each other, taking care to place *units* under *units*, *tens* under *tens*, *hundreds* under *hundreds*, and so on.

When the numbers are thus written, the *right-hand figure of one number is placed directly under the right-hand figure of the one above it*, thus bringing units under units, tens under tens, etc. Proceed as in the following examples:

28. EXAMPLE.—What is the sum of 131, 222, 21, 2, and 413?

SOLUTION.—

$$\begin{array}{r} 131 \\ 222 \\ 21 \\ 2 \\ 413 \\ \hline \textit{sum}\ 789 \end{array}$$ Ans.

EXPLANATION. — After placing the numbers in proper order, begin at the bottom of the right-hand, or units, column, and add, mentally repeating the different sums. Thus, three and two are five and one are six and two are eight and one are nine, the sum of the numbers in units column. Place the 9 directly beneath as the first, or units, figure in the sum.

The sum of the numbers in the next, or tens, column equals 8 tens, which is the second, or tens, figure in the sum.

The sum of the numbers in the next, or hundreds, column equals 7 hundreds, which is the third, or hundreds, figure in the sum.

The sum, or answer, is 789.

29. EXAMPLE.—What is the sum of 425, 36, 9,215, 4, and 907?

SOLUTION.—

$$\begin{array}{r} 425 \\ 36 \\ 9215 \\ 4 \\ 907 \\ \hline 27 \\ 60 \\ 1500 \\ 9000 \\ \hline \textit{sum}\ 10587 \end{array}$$ Ans.

EXPLANATION.—The sum of the numbers in the first, or

units, column is seven and four are eleven and five are sixteen and six are twenty-two and five are twenty-seven, or 27 units; i. e., two tens and seven units. Write 27 as shown. The sum of the numbers in the second, or tens, column is six tens, or 60. Write 60 underneath 27, as shown. The sum of the numbers in the third, or hundreds, column is 15 hundreds, or 1,500. Write 1,500 under the two preceding results as shown. There is only one number in the fourth, or thousands, column, 9, which represents 9,000. Write 9,000 under the three preceding results. Adding these four results, the sum is 10,587, which is the sum of 425, 36, 9,215, 4, and 907.

NOTE.—It frequently happens when adding a long column of figures, that the sum of two numbers, one of which does not occur in the addition table, is required. Thus, in the first column above, the sum of 16 and 6 was required. We know from the table that $6+6=12$; hence, the first figure of the sum is 2. Now, the sum of any number less than 20 and of any number less than 10 must be less than 30, since $20+10=30$; therefore, the sum is 22. Consequently, in cases of this kind, add the first figure of the larger number to the smaller number, and if the result is greater than 9, increase the second figure of the larger number by 1. Thus, $44+7=?$ $4+7=11$; hence, $44+7=51$.

30. The addition may also be performed as follows:

	425
	36
	9215
	4
	907
sum	10587 Ans.

EXPLANATION.—The sum of the numbers in units column is 27 units, or 2 tens and 7 units. Write the 7 units as the first, or right-hand, figure in the sum. Reserve the two tens and add them to the figures in tens column. The sum of the figures in the tens column, plus the 2 tens reserved and carried from the units column, is 8, which is written down as the second figure in the sum. There is nothing to carry to the next column, because 8 is less than 10. The sum of the numbers in the next column is 15 hundreds, or 1 thousand and 5 hundreds. Write down the 5 as the third, or hundreds, figure in the sum and carry the 1 to the next

column. $1+9=10$, which is written down at the left of the other figures.

The second method saves space and figures, but the first is to be preferred when adding a long column.

31. EXAMPLE.—Add the numbers in the column below:

SOLUTION.—

	890
	82
	90
	393
	281
	80
	770
	83
	492
	80
	383
	84
	191
sum	3899 Ans.

EXPLANATION.—The sum of the digits in the first column equals 19 units, or 1 ten and 9 units. Write down the 9 and carry 1 to the next column. The sum of the digits in the second column $+1$ is 109 tens, or 10 hundreds and 9 tens. Write down the 9 and carry the 10 to the next column. The sum of the digits in this column plus the 10 reserved is 38.

The entire sum is 3,899.

32. **Rule.—I.** *Begin at the right, add each column separately, and write the sum, if it be only one figure, under the column added.*

II. *If the sum of any column consists of two or more figures, put the right-hand figure of the sum under that column and add the remaining figure or figures to the next column.*

33. **Proof.**—*To prove addition, add each column from top to bottom. If you obtain the same result as by adding from bottom to top, the work is probably correct.*

EXAMPLES FOR PRACTICE.

34. Find the sum of:

(*a*)	104 + 203 + 613 + 214.	(*a*)	1,134.
(*b*)	1,875 + 3,143 + 5,826 + 10,832.	(*b*)	21,676.
(*c*)	4,865 + 2,145 + 8,173 + 40,084.	(*c*)	55,267.
(*d*)	14,204 + 8,173 + 1,065 + 10,042.	(*d*)	33,484.
(*e*)	10,832 + 4,145 + 3,133 + 5,872.	(*e*)	23,982.
(*f*)	214 + 1,231 + 141 + 5,000.	(*f*)	6,586.
(*g*)	123 + 104 + 425 + 126 + 327.	(*g*)	1,105.
(*h*)	6,354 + 2,145 + 2,042 + 1,111 + 3,333.	(*h*)	14,985.

Ans.

SUBTRACTION.

35. In arithmetic, **subtraction** is the process of finding how much greater one number is than another.

The greater of the two numbers is called the **minuend.**

The smaller of the two numbers is called the **subtrahend.**

The number left after subtracting the subtrahend from the minuend is called the **difference,** or **remainder.**

36. The sign of subtraction is −. It is read *minus*, and means *less*. Thus, 12 − 7 is read *12 minus 7*, and means that 7 is to be taken from 12.

37. EXAMPLE.—From 7,568 take 3,425.

SOLUTION.—

minuend	7 5 6 8	
subtrahend	3 4 2 5	
remainder	4 1 4 3	Ans.

EXPLANATION.—Begin at the right-hand, or units, column and subtract in succession each figure in the subtrahend from the one directly above it in the minuend, and write the remainders below the line. The result is the entire remainder.

38. When there are more figures in the minuend than in the subtrahend, and when some figures in the minuend are *less* than the figures directly under them in the subtrahend, proceed as in the following example:

EXAMPLE.—From 8,453 take 844.

SOLUTION.—

minuend	8 4 5 3	
subtrahend	8 4 4	
remainder	7 6 0 9	Ans.

EXPLANATION.—Begin at the right-hand, or units, column to subtract. We cannot take 4 from 3, and must, therefore, borrow 1 from 5 in tens column and annex it to the 3 in units column. The 1 ten = 10 units, which added to the 3 in units column = 13 units. 4 from 13 = 9, the first, or units, figure in the remainder.

Since we borrowed 1 from the 5, only 4 remains; 4 from 4 = 0, the second, or tens, figure. We cannot take 8 from 4, and must, therefore, borrow 1 from 8 in thousands column. Since 1 thousand = 10 hundreds, 10 hundreds + 4 hundreds = 14 hundreds, and 8 from 14 = 6, the third, or hundreds, figure in the remainder.

Since we borrowed 1 from 8, only 7 remains, from which there is nothing to subtract; therefore, 7 is the next figure in the remainder, or answer.

The operation of borrowing is performed by mentally placing 1 before the figure following the one from which it is borrowed. In the above example the 1 borrowed from 5 is placed before 3, making it 13, from which we subtract 4. The 1 borrowed from 8 is placed before 4, making 14, from which 8 is taken.

39. EXAMPLE.—Find the difference between 10,000 and 8,763.

SOLUTION.—

minuend	10000
subtrahend	8763
remainder	1237 Ans.

EXPLANATION.—In the above example we borrow 1 from the second column and place it before 0, making 10; 3 from 10 = 7. In the same way we borrow 1 and place it before the next cipher, making 10; but as we have borrowed 1 from this column and have taken it to the units column, only 9 remains from which to subtract 6; 6 from 9 = 3. For the same reason we subtract 7 from 9 and 8 from 9 for the next two figures, and obtain a total remainder of 1,237.

40. Rule.—*Place the subtrahend (or smaller number) under the minuend (or larger number), in the same manner as for addition, and proceed as in Arts.* **37, 38,** *and* **39.**

41. Proof.—*To prove an example in subtraction, add the subtrahend and the remainder. The sum should equal the minuend. If it does not, a mistake has been made, and the work should be done over.*

Proof of the above example:

subtrahend	8763
remainder	1237
minuend	10000

EXAMPLES FOR PRACTICE.

42. From:

(*a*)	94,278 take 62,574.	(*a*)	31,704.
(*b*)	53,714 take 25,824.	(*b*)	27,890.
(*c*)	71,832 take 58,109.	(*c*)	13,723.
(*d*)	20,804 take 10,408.	(*d*)	10,396.
(*e*)	310,465 take 102,141.	(*e*)	208,324.
(*f*)	(81,043 + 1,041) take 14,831.	(*f*)	67,253.
(*g*)	(20,482 + 18,216) take 21,214.	(*g*)	17,484.
(*h*)	(2,040 + 1,213 + 542) take 3,791.	(*h*)	4.

Ans.

MULTIPLICATION.

43. To **multiply** a number is to *add* it to itself a certain number of times.

44. Multiplication is the process of multiplying one number by another.

The number thus added to itself, or the number to be multiplied, is called the **multiplicand.**

The number which shows how many times the *multiplicand* is to be taken, or the number by which we *multiply*, is called the **multiplier.**

The result obtained by multiplying is called the **product.**

45. The sign of multiplication is $\times$. It is read *times* or *multiplied by*. Thus, 9×6 is read *9 times 6*, or *9 multiplied by 6*.

46. It matters not in what order the numbers to be multiplied together are placed. Thus, 6×9 is the same as 9×6.

47. In the following table, the product of any two numbers (neither of which exceeds 12) may be found:

1 times 1 is 1	2 times 1 is 2	3 times 1 is 3
1 times 2 is 2	2 times 2 is 4	3 times 2 is 6
1 times 3 is 3	2 times 3 is 6	3 times 3 is 9
1 times 4 is 4	2 times 4 is 8	3 times 4 is 12
1 times 5 is 5	2 times 5 is 10	3 times 5 is 15
1 times 6 is 6	2 times 6 is 12	3 times 6 is 18
1 times 7 is 7	2 times 7 is 14	3 times 7 is 21
1 times 8 is 8	2 times 8 is 16	3 times 8 is 24
1 times 9 is 9	2 times 9 is 18	3 times 9 is 27
1 times 10 is 10	2 times 10 is 20	3 times 10 is 30
1 times 11 is 11	2 times 11 is 22	3 times 11 is 33
1 times 12 is 12	2 times 12 is 24	3 times 12 is 36
4 times 1 is 4	5 times 1 is 5	6 times 1 is 6
4 times 2 is 8	5 times 2 is 10	6 times 2 is 12
4 times 3 is 12	5 times 3 is 15	6 times 3 is 18
4 times 4 is 16	5 times 4 is 20	6 times 4 is 24
4 times 5 is 20	5 times 5 is 25	6 times 5 is 30
4 times 6 is 24	5 times 6 is 30	6 times 6 is 36
4 times 7 is 28	5 times 7 is 35	6 times 7 is 42
4 times 8 is 32	5 times 8 is 40	6 times 8 is 48
4 times 9 is 36	5 times 9 is 45	6 times 9 is 54
4 times 10 is 40	5 times 10 is 50	6 times 10 is 60
4 times 11 is 44	5 times 11 is 55	6 times 11 is 66
4 times 12 is 48	5 times 12 is 60	6 times 12 is 72
7 times 1 is 7	8 times 1 is 8	9 times 1 is 9
7 times 2 is 14	8 times 2 is 16	9 times 2 is 18
7 times 3 is 21	8 times 3 is 24	9 times 3 is 27
7 times 4 is 28	8 times 4 is 32	9 times 4 is 36
7 times 5 is 35	8 times 5 is 40	9 times 5 is 45
7 times 6 is 42	8 times 6 is 48	9 times 6 is 54
7 times 7 is 49	8 times 7 is 56	9 times 7 is 63
7 times 8 is 56	8 times 8 is 64	9 times 8 is 72
7 times 9 is 63	8 times 9 is 72	9 times 9 is 81
7 times 10 is 70	8 times 10 is 80	9 times 10 is 90
7 times 11 is 77	8 times 11 is 88	9 times 11 is 99
7 times 12 is 84	8 times 12 is 96	9 times 12 is 108
10 times 1 is 10	11 times 1 is 11	12 times 1 is 12
10 times 2 is 20	11 times 2 is 22	12 times 2 is 24
10 times 3 is 30	11 times 3 is 33	12 times 3 is 36
10 times 4 is 40	11 times 4 is 44	12 times 4 is 48
10 times 5 is 50	11 times 5 is 55	12 times 5 is 60
10 times 6 is 60	11 times 6 is 66	12 times 6 is 72
10 times 7 is 70	11 times 7 is 77	12 times 7 is 84
10 times 8 is 80	11 times 8 is 88	12 times 8 is 96
10 times 9 is 90	11 times 9 is 99	12 times 9 is 108
10 times 10 is 100	11 times 10 is 110	12 times 10 is 120
10 times 11 is 110	11 times 11 is 121	12 times 11 is 132
10 times 12 is 120	11 times 12 is 132	12 times 12 is 144

This table should be carefully committed to memory.

Since 0 has no value, the product of 0 and any number is 0.

48. To multiply a number by one figure only:

EXAMPLE.—Multiply 425 by 5.

SOLUTION.—

multiplicand	4 2 5	
multiplier	5	
product	2 1 2 5	Ans.

EXPLANATION.—For convenience, the multiplier is generally written under the right-hand figure of the multiplicand. On looking in the multiplication table, we see that 5×5 are 25. Multiplying the first figure at the right of the multiplicand, or 5, by the multiplier, 5, it is seen that 5 times 5 units are 25 units, or 2 tens and 5 units. Write the 5 units in units place in the product, and reserve the 2 tens to add to the product of tens. Looking in the multiplication table again, we see that 5×2 are 10. Multiplying the second figure of the multiplicand by the multiplier, 5, we see that 5 times 2 tens are 10 tens, and 10 tens plus the 2 tens reserved are 12 tens, or 1 hundred plus 2 tens. Write the 2 tens in tens place, and reserve the 1 hundred to add to the product of hundreds. Again, we see by the multiplication table that 5×4 are 20. Multiplying the third, or last, figure of the multiplicand by the multiplier, 5, we see that 5 times 4 hundreds are 20 hundreds, and 20 hundreds plus the 1 hundred reserved are 21 hundreds, or 2 thousands and 1 hundred, which we write in thousands and hundreds places, respectively.

Hence, the product is 2,125.

This result is the same as adding 425 five times. Thus,

4 2 5	
4 2 5	
4 2 5	
4 2 5	
4 2 5	
sum 2 1 2 5	Ans.

EXAMPLES FOR PRACTICE.

49. Find the product of:

(*a*) $61{,}483 \times 6$.
(*b*) $12{,}375 \times 5$.
(*c*) $10{,}426 \times 7$.
(*d*) $10{,}835 \times 3$.

Ans. (*a*) 368,898. (*b*) 61,875. (*c*) 72,982. (*d*) 32,505.

(*e*) $98{,}376 \times 4$.
(*f*) $10{,}873 \times 8$.
(*g*) $71{,}543 \times 9$.
(*h*) $218{,}734 \times 2$.

Ans. (*e*) 393,504. (*f*) 86,984. (*g*) 643,887. (*h*) 437,468.

50. To multiply a number by two or more figures:

EXAMPLE.—Multiply 475 by 234.

SOLUTION.—

multiplicand	475
multiplier	234
	1900
	1425
	950
product	111150 Ans.

EXPLANATION.—For convenience, the multiplier is generally written under the multiplicand, placing units under units, tens under tens, etc.

We cannot multiply by 234 at one operation; we must, therefore, multiply by the *parts* and then *add* the **partial products.**

The parts by which we are to multiply are 4 units, 3 tens, and 2 hundreds. 4 times 475 = 1,900, the *first partial product;* 3 times 475 = 1,425, the *second partial product, the right-hand figure of which is written directly under the figure multiplied by*, or 3; 2 times 475 = 950, the *third partial product*, the right-hand figure of which is written directly under the figure multiplied by, or 2.

The sum of these three partial products is 111,150, which is the *entire product.*

51. Rule.—I. *Write the multiplier under the multiplicand, so that units are under units, tens under tens, etc.*

II. *Begin at the right and multiply each figure of the multiplicand by each successive figure of the multiplier, placing the right-hand figure of each partial product directly under the figure used as a multiplier.*

III. *The sum of the partial products will equal the required product.*

52. **Proof.**—*Review the work carefully, or multiply the multiplier by the multiplicand; if the results agree, the work is correct.*

53. When there is a *cipher* in the *multiplier*, multiply by it the same as with the other figures. Thus,

(*a*)	(*b*)	(*c*)	(*d*)
0	2	15	708
× 0	× 0	× 0	× 0
0 Ans.	0 Ans.	00 Ans.	000 Ans.

(*e*)	(*f*)	(*g*)
3114	4008	31264
203	305	1002
9342	20040	62528
0000	0000	00000
6228	12024	00000
632142 Ans.	1222440 Ans.	31264
		31326528 Ans.

When multiplying by a number containing a cipher, the work may be shortened by writing the first cipher of the partial product, then multiplying by the next figure of the multiplier and writing the partial product alongside of the cipher. Thus, examples (*e*) and (*g*) above might have been solved in the following manner:

3114	31264
203	1002
9342	62528
62280	3126400
632142 Ans.	31326528 Ans.

EXAMPLES FOR PRACTICE.

54. Find the product of:

(*a*) 3,842 × 26.
(*b*) 3,716 × 45.
(*c*) 1,817 × 124.
(*d*) 675 × 38.
(*e*) 1,875 × 33.
(*f*) 4,836 × 47.
(*g*) 5,682 × 543.
(*h*) 3,257 × 246.

Ans. (*a*) 99,892. (*b*) 167,220. (*c*) 225,308. (*d*) 25,650. (*e*) 61,875. (*f*) 227,292. (*g*) 3,085,326. (*h*) 801,222.

(*i*)	$2{,}875 \times 302$.	Ans. (*i*)	868,250.
(*j*)	$17{,}819 \times 1{,}004$.	(*j*)	17,890,276.
(*k*)	$38{,}674 \times 205$.	(*k*)	7,928,170.
(*l*)	$18{,}304 \times 100$.	(*l*)	1,830,400.
(*m*)	$7{,}834 \times 10$.	(*m*)	78,340
(*n*)	$87{,}543 \times 1{,}000$.	(*n*)	87,543,000.
(*o*)	$48{,}763 \times 100$.	(*o*)	4,876,300.

DIVISION.

55. **Division** is the process of finding how many times one number is contained in another of the same kind.

The number to be *divided* is called the **dividend.**

The number by which we *divide* is called the **divisor.**

The number which *shows* how many times the *divisor is contained in the dividend* is called the **quotient.**

56. The sign of division is $\div$. It is read *divided by.* $54 \div 9$ is read *54 divided by 9.* Another way to write 54 divided by 9 is $\frac{54}{9}$. Thus, $54 \div 9 = 6$, or $\frac{54}{9} = 6$.

In both of these cases, 54 is the *dividend* and 9 is the *divisor.*

Division is the reverse of multiplication.

57. **To divide when the divisor consists of but one figure,** proceed as in the following example:

EXAMPLE.—What is the quotient of $875 \div 7$?

divisor dividend quotient

SOLUTION.—

```
          7 ) 8 7 5 ( 1 2 5  Ans.
              7
              ---
              1 7
              1 4
              ---
                3 5
                3 5
                ---
remainder         0
```

EXPLANATION.— 7 is contained in 8 hundreds, 1 hundred times. Place the 1 as the first, or left-hand, figure of the quotient. Multiply the divisor, 7, by the 1 hundred of the

quotient, and place the product, 7 hundreds, under the 8 hundreds in the dividend, and subtract. Beside the remainder, 1, bring down the next, or tens, figure of the dividend, in this case 7, making 17 tens; 7 is contained in 17, 2 times. Write the 2 as the second figure of the quotient. Multiply the divisor, 7, by the 2 in the quotient, and subtract the product from 17. Beside the remainder, 3, bring down the units figure of the dividend, making 35 units. 7 is contained in 35, 5 times, which is placed in the quotient. Multiplying the divisor by the last figure of the quotient, 5 times 7 = 35, which subtracted from 35, under which it is placed, leaves 0. Therefore, the quotient is 125. This method is called **long division.**

58. In **short division,** only the divisor, dividend, and quotient are written, the operations being performed mentally.

dividend

divisor 7) $8^{1}7^{3}5$

quotient 1 2 5 Ans.

The mental operation is as follows: 7 is contained in 8, once and 1 remainder; imagine 1 to be placed before 7, making 17; 7 is contained in 17, 2 times and 3 over; imagine 3 to be placed before 5, making 35; 7 is contained in 35, 5 times. These partial quotients, placed in order as they are found, make the entire quotient 125.

59. If the divisor consists of *two or more* figures, proceed as in the following example:

EXAMPLE.—Divide 2,702,826 by 63.

divisor *dividend* *quotient*

SOLUTION.— 63) 2702826 (42902 Ans.

252

182

126

568

567

126

126

EXPLANATION.—As 63 is not contained in the first two figures, 27, we must use the first three figures, 270. Now, by trial we must find how many times 63 is contained in 270. 6 is contained in the first two figures of 270, 4 times. Place the 4 as the first, or left-hand, figure in the quotient. Multiply the divisor, 63, by 4, and subtract the product, 252, from 270. The remainder is 18, beside which we write the next figure of the dividend, 2, making 182. Now, 6 is contained in the first two figures of 182, 3 times, but on multiplying 63 by 3, we see that the product, 189, is too great, so we try 2 as the second figure of the quotient. Multiplying the divisor, 63, by 2 and subtracting the product, 126, from 182, the remainder is 56, beside which we bring down the next figure of the dividend, making 568. 6 is contained in 56 about 9 times. Multiply the divisor, 63, by 9 and subtract the product, 567, from 568. The remainder is 1, and bringing down the next figure of the dividend, 2, gives 12. As 12 is smaller than 63, we write 0 in the quotient and bring down the next figure, 6, making 126. 63 is contained in 126, 2 times, without a remainder. Therefore, 42,902 is the quotient.

60. Rule.—I. *Write the divisor at the left of the dividend, with a line between them.*

II. *Find how many times the divisor is contained in the lowest number of the left-hand figures of the dividend that will contain it, and write the result at the right of the dividend, with a line between, for the first figure of the quotient.*

III. *Multiply the divisor by this quotient; write the product under the partial dividend used, and subtract, annexing to the remainder the next figure of the dividend. Divide as before, and thus continue until all the figures of the dividend have been used.*

IV. *If any partial dividend will not contain the divisor, write a cipher in the quotient, annex the next figure of the dividend, and proceed as before.*

v. *If there be at last a remainder, write it after the quotient, with the divisor underneath.*

61. Proof.—*Multiply the quotient by the divisor and add the remainder, if there be any, to the product. The result will be the dividend.* Thus,

divisor *dividend* *quotient*

63) 4235 ($67\frac{14}{63}$ Ans.

378

455

441

remainder 14

PROOF.— *quotient* 67

divisor 63

201

402

4221

remainder 14

dividend 4235

EXAMPLES FOR PRACTICE.

62. Divide the following:

(*a*) 126,498 by 58.
(*b*) 3,207,594 by 767.
(*c*) 11,408,202 by 234.
(*d*) 2,100,315 by 581.
(*e*) 969,936 by 4,008.
(*f*) 7,481,888 by 1,021.
(*g*) 1,525,915 by 5,003.
(*h*) 1,646,301 by 381.

Ans. (*a*) 2,181. (*b*) 4,182. (*c*) 48,753. (*d*) 3,615. (*e*) 242. (*f*) 7,328. (*g*) 305. (*h*) 4,321.

CANCELATION.

63. Cancelation is the process of shortening operations in division by casting out equal factors from both dividend and divisor.

64. The **factors** of a number are those numbers, which, when multiplied together, *produce the given number.* Thus, 5 and 3 are the factors of 15, since $5 \times 3 = 15$. Likewise, 8 and 7 are the factors of 56, since $8 \times 7 = 56$.

65. A **prime number** is one which cannot be divided by any number except itself and 1. Thus, 2, 3, 11, 29, etc. are prime numbers.

66. A **prime factor** is any factor that is a prime number.

Any number that is not a prime is called a **composite** number, and may be produced by multiplying together its prime factors. Thus, 60 is a composite number, and is equal to the product of its prime factors, $2 \times 2 \times 3 \times 5$.

67. *Canceling equal factors from both dividend and divisor does not change the quotient.*

The canceling of a factor in *both* dividend and divisor is the same as *dividing them both by the same number*, and this, evidently, *does not change the quotient.*

Write the numbers forming the dividend above a horizontal line, and those forming the divisor below it; then cancel the equal factors.

68. EXAMPLE.—Divide $4 \times 45 \times 60$ by 9×24.

SOLUTION.—Placing the dividend over the divisor, and canceling,

$$\frac{\overset{5}{\cancel{4}} \times \cancel{45} \times \overset{10}{\cancel{60}}}{\underset{6}{\cancel{24}}} = \frac{50}{1} = 50. \quad \text{Ans.}$$

EXPLANATION.—The 4 in the dividend and the 24 in the divisor are both divisible by 4, since 4 divided by 4 equals 1, and 24 divided by 4 equals 6. Cross off the 4 and write the 1 over it; also, cross off the 24 and write the 6 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times 45 \times 60}{9 \times \underset{6}{\cancel{24}}} =$$

60 in the dividend and 6 in the divisor are divisible by 6, since 60 divided by 6 equals 10, and 6 divided by 6 equals 1. Cross off the 60 and write 10 over it; also, cross off the 6 and write 1 under it. Thus,

$$\frac{1 \qquad \overset{10}{\cancel{60}}}{4 \times 45 \times \underset{6}{\cancel{24}}} =$$

Again, 45 in the dividend and 9 in the divisor are divisible by 9, since 45 divided by 9 equals 5, and 9 divided by 9 equals 1. Cross off the 45 and write the 5 over it; also, cross off the 9 and write the 1 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times \overset{5}{\cancel{45}} \times 60}{\underset{1}{\cancel{9}} \times \underset{1}{\cancel{24}}} =$$

Since there are no two remaining numbers (one in the dividend and one in the divisor) divisible by any number except 1, without a remainder, it is impossible to cancel further.

Multiply all the uncanceled numbers in the dividend together and divide their product by the product of all the uncanceled numbers in the divisor. The result will be the quotient. The product of all the uncanceled numbers in the dividend equals $5 \times 1 \times 10 = 50$; the product of all the uncanceled numbers in the divisor equals $1 \times 1 = 1$.

$$\text{Hence, } \frac{\overset{1}{\cancel{4}} \times \overset{5}{\cancel{45}} \times 60}{\underset{1}{\cancel{9}} \times \underset{1}{\cancel{24}}} = \frac{1 \times 5 \times 10}{1 \times 1} = 50. \quad \text{Ans.}$$

69. **Rule.**—I. *Cancel the common factors from both the dividend and the divisor.*

II. *Then divide the product of the remaining factors of the dividend by the product of the remaining factors of the divisor, and the result will be the quotient.*

EXAMPLES FOR PRACTICE.

70. Divide:

(*a*)	$14 \times 18 \times 16 \times 40$ by $7 \times 8 \times 6 \times 5 \times 3$.	Ans.	(*a*) 32.
(*b*)	$3 \times 65 \times 50 \times 100 \times 60$ by $30 \times 60 \times 13 \times 10$.		(*b*) 250.
(*c*)	$8 \times 4 \times 3 \times 9 \times 11$ by $11 \times 9 \times 4 \times 3 \times 8$.		(*c*) 1.
(*d*)	$164 \times 321 \times 6 \times 7 \times 4$ by $82 \times 321 \times 7$.		(*d*) 48.
(*e*)	$50 \times 100 \times 200 \times 72$ by $1{,}000 \times 144 \times 100$.		(*e*) 5.
(*f*)	$48 \times 63 \times 55 \times 49$ by $7 \times 21 \times 11 \times 48$.		(*f*) 105.
(*g*)	$110 \times 150 \times 84 \times 32$ by $11 \times 15 \times 100 \times 64$.		(*g*) 42.
(*h*)	$115 \times 120 \times 400 \times 1{,}000$ by $23 \times 1{,}000 \times 60 \times 800$.		(*h*) 5.

FRACTIONS.

DEFINITIONS.

71. A **fraction** is a *part of a unit*. *One-half*, *one-third*, *two-fifths* are fractions.

72. *Two* numbers are required to express a fraction; one is called the **numerator,** and the other, the **denominator.**

73. The numerator is placed above the denominator, with a line between them, as $\frac{2}{3}$. Here, 3 is the denominator, and shows into how many *equal parts* the *unit*, or *one*, is divided. The numerator, 2, shows how many of these equal parts are taken or considered. The denominator also indicates the names of the parts.

$\frac{1}{2}$ is read one-half.
$\frac{3}{4}$ is read three-fourths.
$\frac{3}{8}$ is read three-eighths.
$\frac{5}{16}$ is read five-sixteenths.
$\frac{29}{47}$ is read twenty-nine forty-sevenths.

74. In the expression "$\frac{3}{4}$ of an apple," the *denominator*, 4, shows that the apple is to be (or has been) cut into 4 *equal parts*, and the *numerator*, 3, shows that *three of these parts*, or *fourths*, are taken or considered.

If each of the *parts*, or *fourths*, of the apple were cut into *two equal pieces*, there would then be twice as many pieces as before, or $4 \times 2 = 8$ pieces in all, one of these pieces would be called one-eighth, and would be expressed in figures as $\frac{1}{8}$. Three of these pieces would be called three-eighths, and written $\frac{3}{8}$. The words three-fourths, three-eighths, five-sixteenths, etc. are abbreviations of three one-fourths, three one-eighths, five one-sixteenths, etc. It is evident that the larger the *denominator*, the greater is the number of parts into which anything is divided; consequently, the parts themselves are smaller, and the value of the fraction is less for the same number of parts taken. In other words, $\frac{7}{9}$, for example, is smaller than $\frac{7}{8}$, because if an object be divided into 9 parts, the parts are smaller than if

the same object had been divided into 8 parts; and, since $\frac{1}{9}$ is smaller than $\frac{1}{8}$, it is clear that 7 one-ninths is a smaller amount than 7 one-eighths. Hence, also, $\frac{3}{8}$ is less than $\frac{3}{4}$.

75. The **value** of a fraction is the numerator divided by the denominator, as $\frac{4}{2} = 2$, $\frac{6}{2} = 3$.

76. The line between the numerator and the denominator means *divided by*, or $\div$.

$\frac{3}{4}$ is equivalent to $3 \div 4$.

$\frac{5}{8}$ is equivalent to $5 \div 8$.

77. The numerator and the denominator of a fraction are called the **terms** of a fraction.

78. The *value* of a fraction whose numerator and denominator are equal is 1.

$\frac{4}{4}$, or four-fourths $= 1$.

$\frac{8}{8}$, or eight-eighths $= 1$.

$\frac{64}{64}$, or sixty-four sixty-fourths $= 1$.

79. A **proper fraction** is a fraction whose numerator is *less* than its denominator. Its value is *less* than 1, as $\frac{3}{4}$, $\frac{5}{8}$, $\frac{1}{16}$.

80. An **improper fraction** is a fraction whose numerator *equals* or is *greater* than the denominator. Its value is 1 or *more than* 1, as $\frac{4}{4}$, $\frac{9}{8}$, $\frac{42}{32}$.

81. A **mixed number** is a whole number and a fraction united. $4\frac{2}{3}$ is a mixed number, and is equivalent to $4 + \frac{2}{3}$. It is read *four and two-thirds.*

REDUCTION OF FRACTIONS.

82. **Reduction of fractions** is the process of changing their form without changing their *value.*

83. *A fraction is reduced to higher terms by multiplying both terms of the fraction by the same number.* Thus, $\frac{3}{4}$ is reduced to $\frac{6}{8}$ by multiplying both terms by 2.

$$\frac{3 \times 2}{4 \times 2} = \frac{6}{8}.$$

The value is not changed, since $\frac{3}{4} = \frac{6}{8}$. For, suppose that an object, say an apple, is divided into 8 equal parts. If these parts be arranged into 4 piles, each containing 2 parts, it is evident each pile will be composed of the same amount of the entire apple as would have been the case had the apple been originally cut into 4 equal parts. Now, if one of these piles (containing 2 parts) be removed, there will be 3 piles left, each containing 2 equal parts, or 6 equal parts in all, i. e., six-eighths. But, since one pile, or one-quarter, was removed, there are three-quarters left. Hence, $\frac{3}{4} = \frac{6}{8}$. The same course of reasoning may be applied to any similar case. Therefore, multiplying both terms of a fraction by the same number does not alter its value.

84. To reduce a fraction to an equal fraction having a given denominator:

EXAMPLE.—Reduce $\frac{7}{8}$ to an equal fraction having 96 for a denominator.

SOLUTION.—Both the numerator and the denominator must be multiplied by the same number in order not to change the value of the fraction. The denominator must be multiplied by some number which will, in this case, make the product 96; this number is evidently $96 \div 8 = 12$, since $8 \times 12 = 96$. Hence, $\frac{7 \times 12}{8 \times 12} = \frac{84}{96}$. Ans.

85. Rule.—*Divide the given denominator by the denominator of the given fraction, and multiply both terms of the fraction by the result.*

EXAMPLE.—Reduce $\frac{3}{4}$ to 100ths.

SOLUTION.— $100 \div 4 = 25$; hence, $\frac{3 \times 25}{4 \times 25} = \frac{75}{100}$. Ans.

86. *A fraction is reduced to lower terms by dividing both terms by the same number.* Thus, $\frac{8}{10}$ is reduced to $\frac{4}{5}$ by dividing both terms by 2.

$$\frac{8 \div 2}{10 \div 2} = \frac{4}{5}.$$

That $\frac{8}{10} = \frac{4}{5}$ is readily seen from the explanation given in Art. **83**; for, multiplying both terms of the fraction $\frac{4}{5}$ by 2, $\frac{4}{5} \times \frac{2}{2} = \frac{8}{10}$, and, if $\frac{4}{5} = \frac{8}{10}$, $\frac{8}{10}$ must equal $\frac{4}{5}$. Hence, dividing both terms of a fraction by the same number does not alter its value.

87. A fraction is reduced to its *lowest terms* when its numerator and denominator cannot both be divided by the *same* number without a remainder; for example, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{11}{24}$, $\frac{8}{15}$.

EXAMPLES FOR PRACTICE.

88. Reduce the following:

(*a*) $\frac{7}{16}$ to 128ths.
(*b*) $\frac{24}{132}$ to its lowest terms.
(*c*) $\frac{64}{1000}$ to its lowest terms.
(*d*) $\frac{5}{7}$ to 49ths.
(*e*) $\frac{13}{16}$ to 10,000ths.

Ans. (*a*) $\frac{56}{128}$. (*b*) $\frac{2}{11}$. (*c*) $\frac{8}{125}$. (*d*) $\frac{35}{49}$. (*e*) $\frac{8125}{10000}$.

89. To reduce a whole number or a mixed number to an improper fraction:

EXAMPLE.—How many fourths in 5?

SOLUTION.—Since there are 4 fourths in 1 ($\frac{4}{4} = 1$), in 5 there will be 5×4 fourths, or 20 fourths; i. e., $5 \times \frac{4}{4} = \frac{20}{4}$. Ans.

EXAMPLE.—Reduce $8\frac{3}{4}$ to an improper fraction.

SOLUTION.— $8 \times \frac{4}{4} = \frac{32}{4}$. $\frac{32}{4} + \frac{3}{4} = \frac{35}{4}$. Ans.

90. Rule.—*Multiply the whole number by the denominator of the fraction, add the numerator to the product and place the denominator under the result. If it is desired to reduce a whole number to a fraction, multiply the whole number by the denominator of the given fraction, and write the result over the denominator.*

EXAMPLES FOR PRACTICE.

91. Reduce to improper fractions:

(*a*) $4\frac{1}{8}$.
(*b*) $5\frac{1}{9}$.
(*c*) $10\frac{2}{10}$.
(*d*) $37\frac{3}{4}$.
(*e*) $50\frac{4}{5}$.
(*f*) Reduce 7 to a fraction whose denominator is 16.

Ans. (*a*) $\frac{33}{8}$. (*b*) $\frac{46}{9}$. (*c*) $\frac{102}{10}$. (*d*) $\frac{151}{4}$. (*e*) $\frac{254}{5}$. (*f*) $\frac{112}{16}$.

92. To reduce an improper fraction to a whole or a mixed number:

EXAMPLE.—Reduce $\frac{21}{4}$ to a mixed number.

SOLUTION.— 4 is contained in 21, 5 times and 1 remaining (see Art. **75**); as this is also divided by 4, its value is $\frac{1}{4}$. Therefore, $5 + \frac{1}{4}$, or $5\frac{1}{4}$, is the number. Ans.

93. **Rule.**—*Divide the numerator by the denominator, the quotient will be the whole number; the remainder, if there be any, will be the numerator of the fractional part of which the denominator is the same as the denominator of the improper fraction.*

EXAMPLES FOR PRACTICE.

94. Reduce to whole or mixed numbers:

(*a*) $\frac{145}{6}$.
(*b*) $\frac{185}{3}$.
(*c*) $\frac{701}{6}$.
(*d*) $\frac{149}{3}$.
(*e*) $\frac{76}{19}$.
(*f*) $\frac{1255}{255}$.

Ans. (*a*) $24\frac{1}{6}$. (*b*) $61\frac{2}{3}$. (*c*) $116\frac{5}{6}$. (*d*) $49\frac{2}{3}$. (*e*) 4. (*f*) 5.

---

95. A **common denominator** of two or more fractions is a number which will contain (i. e., which may be divided by) the denominator of each of the given fractions without a remainder. The **least common denominator** is the least number that will contain each denominator of the given fractions without a remainder.

96. **To find the least common denominator:**

EXAMPLE.—Find the least common denominator of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{9}$, and $\frac{1}{16}$.

SOLUTION.—We first place the denominators in a row, separated by commas.

2) 4, 3, 9, 16
2) 2, 3, 9, 8
3) 1, 3, 9, 4
3) 1, 1, 3, 4
4) 1, 1, 1, 4
1, 1, 1, 1

$2 \times 2 \times 3 \times 3 \times 4 = 144$, the least common denominator. Ans.

EXPLANATION.—Divide each of them by some prime number which will divide at least two of them without a remainder (if possible), bringing down those denominators to the row below which will not contain the divisor without a remainder. Dividing each of the numbers by 2, the second row becomes 2, 3, 9, 8, since 2 will not divide 3 and 9 without a remainder. Dividing again by 2, the result is 1, 3, 9, 4. Dividing the third row by 3, the result, is 1, 1,

3, 4. So continue until the last row contains only 1's. The product of all the divisors, or $2 \times 2 \times 3 \times 3 \times 4 = 144$, is the least common denominator.

97. EXAMPLE.—Find the least common denominator of $\frac{4}{9}$, $\frac{5}{12}$, $\frac{7}{18}$.

SOLUTION.—

3) 9, 12, 18
3) 3, 4, 6
2) 1, 4, 2
2) 1, 2, 1
1, 1, 1

$3 \times 3 \times 2 \times 2 = 36$. Ans.

98. To reduce two or more fractions to fractions having a common denominator:

EXAMPLE.—Reduce $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{1}{2}$ to fractions having a common denominator.

SOLUTION.—The common denominator is a number which will contain 3, 4, and 2. The least common denominator is 12, because it is the smallest number which can be divided by 3, 4, and 2 without a remainder.

$$\frac{2}{3} = \frac{8}{12}, \quad \frac{3}{4} = \frac{9}{12}, \quad \frac{1}{2} = \frac{6}{12}.$$

Reducing $\frac{2}{3}$ (see Art. **84**), 3 is contained in 12, 4 times. By multiplying both numerator and denominator of $\frac{2}{3}$ by 4, we find

$\frac{2 \times 4}{3 \times 4} = \frac{8}{12}$. In the same way we find $\frac{3}{4} = \frac{9}{12}$ and $\frac{1}{2} = \frac{6}{12}$.

99. Rule.—*Divide the common denominator by the denominator of the given fraction, and multiply both terms of the fraction by the quotient.*

EXAMPLES FOR PRACTICE.

100. Reduce to fractions having a common denominator:

(*a*) $\frac{3}{4}$, $\frac{5}{8}$, $\frac{7}{8}$.
(*b*) $\frac{3}{16}$, $\frac{3}{4}$, $\frac{7}{32}$.
(*c*) $\frac{7}{8}$, $\frac{7}{88}$, $\frac{10}{11}$.
(*d*) $\frac{3}{5}$, $\frac{5}{8}$, $\frac{11}{40}$.
(*e*) $\frac{4}{10}$, $\frac{6}{40}$, $\frac{9}{20}$.
(*f*) $\frac{7}{15}$, $\frac{17}{30}$, $\frac{21}{30}$.

Ans.

(*a*) $\frac{6}{8}$, $\frac{5}{8}$, $\frac{7}{8}$.
(*b*) $\frac{6}{32}$, $\frac{24}{32}$, $\frac{7}{32}$.
(*c*) $\frac{77}{88}$, $\frac{7}{88}$, $\frac{80}{88}$.
(*d*) $\frac{24}{40}$, $\frac{25}{40}$, $\frac{11}{40}$.
(*e*) $\frac{16}{40}$, $\frac{6}{40}$, $\frac{18}{40}$.
(*f*) $\frac{14}{30}$, $\frac{17}{30}$, $\frac{21}{30}$.

ADDITION OF FRACTIONS.

101. *Fractions cannot be added unless they have a common denominator.* We cannot add $\frac{3}{4}$ to $\frac{7}{8}$ as they now stand, since the denominators represent parts of different sizes. Fourths cannot be added to eighths.

Suppose we divide an apple into 4 equal parts, and then divide 2 of these parts into 2 equal parts. It is evident that we shall have 2 one-fourths and 4 one-eighths. Now, if we add these parts, the result is $2+4=6$ something. But what is this something? It is not fourths, for 6 fourths are $1\frac{1}{2}$, and we had only 1 apple to begin with; neither is it eighths, for 6 eighths are $\frac{3}{4}$, which is less than 1 apple. By reducing the quarters to eighths, we have $\frac{2}{4}=\frac{4}{8}$, and adding the other 4 eighths, $4+4=8$ eighths. This result is correct, since $\frac{8}{8}=1$. Or we can, in this case, reduce the eighths to quarters. Thus, $\frac{4}{8}=\frac{2}{4}$; whence, adding, $2+2=4$ quarters, a correct result, since $\frac{4}{4}=1$.

Before adding, fractions should be reduced to a common denominator, preferably the *least* common denominator.

102. Example.—Find the sum of $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{8}$.

Solution.—The *least common denominator*, or the *least number* which will contain all the denominators, is 8.

$$\frac{1}{2}=\frac{4}{8},\quad \frac{3}{4}=\frac{6}{8},\quad \text{and}\quad \frac{5}{8}=\frac{5}{8}.$$

Explanation.—As the denominator tells or indicates the *names* of the parts, the numerators only are added, to obtain the total number of *parts* indicated by the denominator. Thus, 4 one-eighths plus 6 one-eighths plus 5 one-eighths $=$

$$\frac{4}{8}+\frac{6}{8}+\frac{5}{8}=\frac{4+6+5}{8}=\frac{15}{8}=1\tfrac{7}{8}.\quad \text{Ans.}$$

103. Example.—What is the sum of $12\frac{3}{4}$, $14\frac{1}{8}$, and $7\frac{5}{16}$?

Solution.—The least common denominator in this case is 16.

$$\begin{aligned} 12\tfrac{3}{4} &= 12\tfrac{12}{16} \\ 14\tfrac{1}{8} &= 14\tfrac{2}{16} \\ 7\tfrac{5}{16} &= 7\tfrac{5}{16} \\ \textit{sum}\quad & 33+\tfrac{27}{16}=33+1\tfrac{11}{16}=34\tfrac{11}{16}. \end{aligned}$$ Ans.

The sum of the fractions $=\frac{27}{16}$ or $1\frac{11}{16}$, which added to the sum of the whole numbers $=34\frac{11}{16}$.

Example.—What is the sum of 17, $13\frac{3}{16}$, $\frac{9}{32}$, and $3\frac{1}{4}$?

Solution.—The least common denominator is 32. $13\frac{3}{16}=13\frac{6}{32}$, $3\frac{1}{4}=3\frac{8}{32}$.

$$\begin{aligned} & 17 \\ & 13\tfrac{6}{32} \\ & \tfrac{9}{32} \\ & 3\tfrac{8}{32} \\ \textit{sum}\quad & 33\tfrac{23}{32}. \end{aligned}$$ Ans.

104. **Rule.—I.** *Reduce the given fractions to fractions having the least common denominator, and write the sum of the numerators over the common denominator.*

II. *When there are mixed numbers and whole numbers, add the fractions first, and if their sum is an improper fraction, reduce it to a mixed number and add the whole number with the other whole numbers.*

EXAMPLES FOR PRACTICE.

105. Find the sum of:

(*a*) $\frac{4}{6}$, $\frac{7}{24}$, $\frac{5}{8}$.
(*b*) $\frac{2}{3}$, $\frac{5}{15}$, $\frac{24}{45}$.
(*c*) $\frac{1}{2}$, $\frac{8}{9}$, $\frac{5}{16}$.
(*d*) $\frac{5}{6}$, $\frac{11}{72}$, $\frac{13}{18}$.
(*e*) $\frac{10}{11}$, $\frac{6}{33}$, $\frac{23}{66}$.
(*f*) $\frac{23}{45}$, $\frac{11}{15}$, $\frac{14}{45}$.
(*g*) $\frac{4}{11}$, $\frac{7}{22}$, $\frac{14}{22}$.
(*h*) $\frac{3}{7}$, $\frac{14}{49}$, $\frac{2}{7}$.

Ans. (*a*) $1\frac{7}{12}$. (*b*) $1\frac{8}{15}$. (*c*) $1\frac{3}{16}$. (*d*) $1\frac{51}{72}$. (*e*) $1\frac{29}{66}$. (*f*) $1\frac{5}{9}$. (*g*) $1\frac{7}{22}$. (*h*) 1.

SUBTRACTION OF FRACTIONS.

106. *Fractions cannot be subtracted without first reducing them to a common denominator.* This can be shown in the same manner as in the case of addition of fractions.

EXAMPLE.—Subtract $\frac{3}{8}$ from $1\frac{13}{16}$.

SOLUTION.—The common denominator is 16.

$$\frac{3}{8} = \frac{6}{16}. \quad 1\frac{13}{16} - \frac{6}{16} = \frac{13-6}{16} = \frac{7}{16}. \text{ Ans.}$$

107. EXAMPLE.—From 7 take $\frac{5}{8}$.

SOLUTION.— $1 = \frac{8}{8}$; therefore, since $7 = 6+1$, $7 = 6 + \frac{8}{8} = 6\frac{8}{8}$, or $6\frac{8}{8} - \frac{5}{8} = 6\frac{3}{8}$. Ans.

108. EXAMPLE.—What is the difference between $17\frac{9}{16}$ and $9\frac{15}{32}$?

SOLUTION.—The common denominator of the fractions is 32. $17\frac{9}{16} = 17\frac{18}{32}$.

minuend	$17\frac{18}{32}$
subtrahend	$9\frac{15}{32}$
difference	$8\frac{3}{32}$. Ans.

109. EXAMPLE.—From $9\frac{1}{4}$ take $4\frac{7}{16}$.

SOLUTION.—The common denominator of the fractions is 16. $9\frac{1}{4} = 9\frac{4}{16}$.

minuend	$9\frac{4}{16}$ or $8\frac{20}{16}$	
subtrahend	$4\frac{7}{16}$	$4\frac{7}{16}$
difference	$4\frac{13}{16}$	$4\frac{13}{16}$. Ans.

EXPLANATION.—As the fraction in the subtrahend is greater than the fraction in the minuend, it cannot be subtracted; therefore, *borrow* 1, or $\frac{16}{16}$, from the 9 in the minuend and add it to the $\frac{4}{16}$; $\frac{4}{16} + \frac{16}{16} = \frac{20}{16}$. $\frac{7}{16}$ from $\frac{20}{16} = \frac{13}{16}$. Since 1 was borrowed from 9, 8 remains; 4 from 8 = 4; $4 + \frac{13}{16} = 4\frac{13}{16}$.

110. EXAMPLE.—From 9 take $8\frac{3}{16}$.

SOLUTION.—

minuend	9	or $8\frac{16}{16}$
subtrahend	$8\frac{3}{16}$	$8\frac{3}{16}$
difference	$\frac{13}{16}$	$\frac{13}{16}$. Ans.

EXPLANATION.—As there is no fraction in the minuend from which to take the fraction in the subtrahend, borrow 1, or $\frac{16}{16}$, from 9. $\frac{3}{16}$ from $\frac{16}{16} = \frac{13}{16}$. Since 1 was borrowed from 9, only 8 is left. 8 from 8 = 0.

111. Rule.—I. *Reduce the fractions to fractions having a common denominator. Subtract one numerator from the other and place the remainder over the common denominator.*

II. *When there are mixed numbers, subtract the fractions and whole numbers separately, and place the remainders side by side.*

III. *When the fraction in the subtrahend is greater than the fraction in the minuend, borrow 1 from the whole number in the minuend and add it to the fraction in the minuend, from which subtract the fraction in the subtrahend.*

IV. *When the minuend is a whole number, borrow 1; reduce it to a fraction whose denominator is the same as the denominator of the fraction in the subtrahend, and place it over that fraction for subtraction.*

EXAMPLES FOR PRACTICE.

112. Subtract:

(*a*) $\frac{10}{24}$ from $\frac{11}{12}$.
(*b*) $\frac{7}{14}$ from $\frac{17}{28}$.
(*c*) $\frac{4}{80}$ from $\frac{5}{10}$.
(*d*) $\frac{15}{85}$ from $\frac{45}{70}$.
(*e*) $\frac{15}{16}$ from $\frac{57}{48}$.
(*f*) $13\frac{1}{4}$ from $30\frac{1}{2}$.
(*g*) $12\frac{1}{8}$ from 27.
(*h*) $5\frac{1}{4}$ from 30.

Ans. (*a*) $\frac{1}{2}$. (*b*) $\frac{3}{28}$. (*c*) $\frac{1}{80}$. (*d*) $\frac{3}{14}$. (*e*) $\frac{1}{4}$. (*f*) $17\frac{1}{4}$. (*g*) $14\frac{7}{8}$. (*h*) $24\frac{1}{4}$.

MULTIPLICATION OF FRACTIONS.

113. *In multiplication of fractions it is not necessary to reduce the fractions to fractions having a common denominator.*

114. *Multiplying the numerator or dividing the denominator multiplies the fraction.*

EXAMPLE.—Multiply $\frac{3}{4}$ by 4.

SOLUTION.— $\frac{3}{4} \times 4 = \frac{3 \times 4}{4} = \frac{12}{4} = 3$. Ans.

Or, $\frac{3}{4} \times 4 = \frac{3}{4 \div 4} = \frac{3}{1} = 3$. Ans.

The word "of," when placed between two fractions, or between a fraction and a whole number, means the same as $\times$, or times. Thus,

$$\frac{3}{4} \text{ of } 4 = \frac{3}{4} \times 4 = 3.$$

$$\frac{1}{8} \text{ of } \frac{5}{16} = \frac{1}{8} \times \frac{5}{16} = \frac{5}{128}.$$

EXAMPLE.—Multiply $\frac{3}{8}$ by 2.

SOLUTION.— $2 \times \frac{3}{8} = \frac{3 \times 2}{8} = \frac{6}{8} = \frac{3}{4}$. Ans.

Or, $2 \times \frac{3}{8} = \frac{3}{8 \div 2} = \frac{3}{4}$. Ans.

115. EXAMPLE.—What is the product of $\frac{4}{16}$ and $\frac{7}{8}$?

SOLUTION.— $\frac{4}{16} \times \frac{7}{8} = \frac{4 \times 7}{16 \times 8} = \frac{28}{128} = \frac{7}{32}$. Ans.

Or, by cancelation, $\frac{\cancel{4} \times 7}{\cancel{16} \times 8} = \frac{7}{4 \times 8} = \frac{7}{32}$. Ans.

116. EXAMPLE.—What is $\frac{4}{8}$ of $\frac{3}{4}$ of $\frac{16}{32}$?

SOLUTION.— $\frac{\cancel{4} \times 3 \times \cancel{16}}{8 \times \cancel{4} \times \cancel{32}} = \frac{3}{8 \times 2} = \frac{3}{16}$. Ans.

117. EXAMPLE.—What is the product of $9\frac{3}{4}$ and $5\frac{5}{8}$?

SOLUTION.— $9\frac{3}{4} = \frac{39}{4}$; $5\frac{5}{8} = \frac{45}{8}$.

$$\frac{39}{4} \times \frac{45}{8} = \frac{39 \times 45}{4 \times 8} = \frac{1755}{32} = 54\frac{27}{32}. \quad \text{Ans.}$$

118. EXAMPLE.—Multiply $15\frac{7}{8}$ by 3.

SOLUTION.—

$$\begin{array}{r} 15\frac{7}{8} \\ \underline{\quad 3} \\ 47\frac{5}{8} \end{array} \quad \text{or} \quad \begin{array}{r} 15\frac{7}{8} \\ \underline{\quad 3} \\ \end{array}$$

$45 + \frac{21}{8} = 45 + 2\frac{5}{8} = 47\frac{5}{8}$. Ans.

119. Rule.—I. *Divide the product of the numerators by the product of the denominators. All factors common to the numerators and denominators should first be cast out by cancelation.*

II. *To multiply one mixed number by another, reduce them both to improper fractions.*

III. *To multiply a mixed number by a whole number, first multiply the fractional part by the multiplier, and if the product is an improper fraction, reduce it to a mixed number and add the whole-number part to the product of the multiplier and the whole number.*

EXAMPLES FOR PRACTICE.

120. Find the product of:

(*a*) $7 \times \frac{3}{19}$.
(*b*) $14 \times \frac{5}{16}$.
(*c*) $\frac{21}{32} \times \frac{5}{14}$.
(*d*) $\frac{16}{27} \times 4$.
(*e*) $\frac{19}{18} \times 7$.
(*f*) $17\frac{18}{21} \times 7$.
(*g*) $\frac{105}{224} \times 32$.
(*h*) $\frac{15}{28} \times 14$.

Ans. (*a*) $1\frac{2}{19}$. (*b*) $4\frac{3}{8}$. (*c*) $\frac{15}{64}$. (*d*) $2\frac{10}{27}$. (*e*) $7\frac{7}{18}$. (*f*) 125. (*g*) 15. (*h*) $7\frac{1}{2}$.

DIVISION OF FRACTIONS.

121. *In division of fractions it is not necessary to reduce the fractions to fractions having a common denominator.*

122. *Dividing the numerator or multiplying the denominator divides the fraction.*

EXAMPLE.—Divide $\frac{6}{8}$ by 3.

SOLUTION.—When *dividing the numerator*, we have

$$\frac{6}{8} \div 3 = \frac{6 \div 3}{8} = \frac{2}{8} = \frac{1}{4}. \quad \text{Ans.}$$

When *multiplying the denominator*, we have

$$\tfrac{6}{8} \div 3 = \frac{6}{8 \times 3} = \tfrac{6}{24} = \tfrac{1}{4}. \quad \text{Ans.}$$

EXAMPLE.—Divide $\frac{3}{16}$ by 2.

SOLUTION.— $$\tfrac{3}{16} \div 2 = \frac{3}{16 \times 2} = \tfrac{3}{32}. \quad \text{Ans.}$$

EXAMPLE.—Divide $\frac{14}{32}$ by 7.

SOLUTION.— $$\tfrac{14}{32} \div 7 = \frac{14 \div 7}{32} = \tfrac{2}{32} = \tfrac{1}{16}. \quad \text{Ans.}$$

123. To **invert** a fraction is to *turn it upside down;* that is, make the numerator and denominator *change places.*

Invert $\frac{3}{4}$ and it becomes $\frac{4}{3}$.

124. EXAMPLE.—Divide $\frac{9}{16}$ by $\frac{3}{16}$.

SOLUTION.—1. The fraction $\frac{3}{16}$ is contained in $\frac{9}{16}$, 3 times, for the denominators are the same, and one numerator is contained in the other 3 times. 2. If we now invert the divisor, $\frac{3}{16}$, and multiply, the solution is

$$\frac{9}{16} \times \frac{16}{3} = \frac{\overset{3}{\cancel{9}} \times \cancel{16}}{\cancel{16} \times \cancel{3}} = 3. \quad \text{Ans.}$$

This brings the same quotient as in the first case.

125. EXAMPLE.—Divide $\frac{3}{8}$ by $\frac{1}{4}$.

SOLUTION.—We cannot divide $\frac{3}{8}$ by $\frac{1}{4}$, as in the first case above, for the denominators are not the same; therefore, we must solve as in the second case.

$$\tfrac{3}{8} \div \tfrac{1}{4} = \tfrac{3}{8} \times \tfrac{4}{1} = \frac{3 \times \overset{1}{\cancel{4}}}{\underset{2}{\cancel{8}} \times 1} = \frac{3}{2} \text{ or } 1\tfrac{1}{2}. \quad \text{Ans.}$$

126. EXAMPLE.—Divide 5 by $\frac{10}{16}$.

SOLUTION.— $\frac{10}{16}$ inverted becomes $\frac{16}{10}$.

$$5 \times \frac{16}{10} = \frac{\overset{8}{\cancel{5}} \times \cancel{16}}{\cancel{10}} = 8. \quad \text{Ans.}$$

127. EXAMPLE.—How many times is $3\frac{3}{4}$ contained in $7\frac{7}{16}$?

SOLUTION.— $3\frac{3}{4} = \frac{15}{4}$; $7\frac{7}{16} = \frac{119}{16}$.

$\frac{15}{4}$ inverted equals $\frac{4}{15}$.

$$\frac{119}{16} \times \frac{4}{15} = \frac{119 \times \overset{1}{\cancel{4}}}{\underset{4}{\cancel{16}} \times 15} = \frac{119}{60} = 1\tfrac{59}{60}. \quad \text{Ans.}$$

128. **Rule.**—*Invert the divisor and proceed as in multiplication.*

129. We have learned that a line placed between two numbers indicates that the number above the line is to be divided by the number below it. Thus, $\frac{18}{3}$ shows that 18 is to be divided by 3. This is also true if a fraction or a fractional expression be placed above or below a line.

$\frac{9}{\frac{3}{8}}$ means that 9 is to be divided by $\frac{3}{8}$; $\frac{3\times 7}{\frac{8+4}{16}}$ means that 3×7 is to be divided by the value of $\frac{8+4}{16}$.

$\frac{\frac{1}{4}}{\frac{3}{8}}$ is the same as $\frac{1}{4}\div\frac{3}{8}$.

It will be noticed that there is a heavy line between the 9 and the $\frac{3}{8}$. This is necessary, since otherwise there would be nothing to show as to whether 9 was to be divided by $\frac{3}{8}$, or $\frac{9}{3}$ was to be divided by 8. Whenever a heavy line is used, as shown here, it indicates that *all above the line is to be divided by all below it.*

EXAMPLES FOR PRACTICE.

130. Divide:

(*a*) 15 by $6\frac{3}{7}$.
(*b*) 30 by $\frac{6}{8}$.
(*c*) 172 by $\frac{4}{5}$.
(*d*) $\frac{14}{18}$ by $1\frac{7}{16}$.
(*e*) $\frac{103}{6}$ by $14\frac{2}{3}$.
(*f*) $\frac{142}{27}$ by $17\frac{1}{9}$.
(*g*) $\frac{14}{18}$ by $\frac{145}{72}$.
(*h*) $\frac{128}{18}$ by $72\frac{1}{8}$.

Ans. (*a*) $2\frac{1}{3}$. (*b*) 40. (*c*) 215. (*d*) $\frac{112}{207}$. (*e*) $1\frac{15}{88}$. (*f*) $\frac{71}{231}$. (*g*) $\frac{56}{145}$. (*h*) $\frac{64}{651}$.

131. Whenever an expression like one of the three following ones is obtained, it may always be simplified by transposing the denominator from *above* to *below* the line, or from *below* to *above*, as the case may be, taking care, however, to indicate that the denominator when so transferred is a multiplier.

1. $\frac{\frac{3}{4}}{9} = \frac{3}{9\times 4} = \frac{3}{36} = \frac{1}{12}$; for, regarding the fraction above the heavy line as the numerator of a fraction whose denominator is 9, $\frac{\frac{3}{4}\times 4}{9\times 4} = \frac{3}{9\times 4}$, as before.

2. $\frac{9}{\frac{3}{4}} = \frac{9\times4}{3} = 12$. The proof is the same as in the first case.

3. $\frac{\frac{5}{9}}{\frac{3}{4}} = \frac{5\times4}{3\times9} = \frac{20}{27}$. For, regarding $\frac{5}{9}$ as the numerator of a fraction whose denominator is $\frac{3}{4}$, $\frac{\frac{5}{9}\times9}{\frac{3}{4}\times9} = \frac{5}{\frac{3\times9}{4}}$; and $\frac{\frac{5}{\frac{3\times9}{4}}\times4}{4} = \frac{5\times4}{3\times9} = \frac{20}{27}$, as above.

This principle may be used to great advantage in cases like $\frac{\frac{1}{4}\times310\times\frac{27}{12}\times72}{40\times4\frac{1}{2}\times5\frac{1}{6}}$. Reducing the mixed numbers to fractions, the expression becomes $\frac{\frac{1}{4}\times310\times\frac{27}{12}\times72}{40\times\frac{9}{2}\times\frac{31}{6}}$. Now transferring the denominators of the fractions and canceling,

$$\frac{1\times310\times27\times72\times2\times6}{40\times9\times31\times4\times12} = \frac{1\times\cancel{310}^{10}\times\cancel{27}^{3}\times\cancel{72}^{\cancel{6}\,3}\times\cancel{2}^{2}\times\cancel{6}^{3}}{\cancel{40}^{\cancel{4}\,10}\times\cancel{9}^{3}\times\cancel{31}\times\cancel{4}^{2}\times\cancel{12}^{\cancel{6}\,3}}$$

$$= \frac{27}{2} = 13\tfrac{1}{2}.$$

Greater exactness in results can usually be obtained by using this principle than by reducing the fractions to decimals. The principle, however, should not be employed if a sign of addition or subtraction occurs either above or below the dividing line.

DECIMALS.

132. **Decimals** are *tenth* fractions; that is, the parts of a unit are expressed on the scale of ten, as *tenths*, *hundredths*, *thousandths*, etc.

133. The *denominator* which is always ten or a multiple of ten, as 10, 100, 1,000, etc., is not expressed, as it would

be in common fractions, by writing it under the *numerator* with a line between them, as $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, but is expressed by placing a *period* (.), which is called a **decimal point,** to the *left* of the *figures of the numerator*, so as to indicate that the number on the right is the numerator of a fraction whose denominator is 10, 100, 1,000, etc.

134. The *reading* of a decimal number depends upon the number of decimal places in it, or the number of figures to the right of the decimal point.

One decimal place expresses *tenths.*
Two decimal places express *hundredths.*
Three decimal places express *thousandths.*
Four decimal places express *ten-thousandths.*
Five decimal places express *hundred-thousandths.*
Six decimal places express *millionths.*

Thus:

.3	=	$\frac{3}{10}$	= 3 tenths.
.03	=	$\frac{3}{100}$	= 3 hundredths.
.003	=	$\frac{3}{1000}$	= 3 thousandths.
.0003	=	$\frac{3}{10000}$	= 3 ten-thousandths.
.00003	=	$\frac{3}{100000}$	= 3 hundred-thousandths.
.000003	=	$\frac{3}{1000000}$	= 3 millionths.

We see in the above that the *number of decimal places in a decimal equals the number of ciphers to the right of the figure 1 in the denominator of its equivalent fraction.* This fact kept in mind will be of much assistance in reading and writing decimals.

Whatever may be written to the *left* of a decimal point is a whole number. The *decimal point merely separates the fraction on the right from the whole number on the left.*

When a whole number and decimal are written together, the expression is a *mixed number.* Thus, 8.12 and 17.25 are mixed numbers.

The relation of decimals and whole numbers to each other is clearly shown by the following table:

hundreds of millions.	tens of millions.	millions.	hundreds of thousands.	tens of thousands.	thousands.	hundreds.	tens.	units.	decimal point.	tenths.	hundredths.	thousandths.	ten-thousandths.	hundred-thousandths.	millionths.	ten-millionths.	hundred-millionths.
9	8	7	6	5	4	3	2	1	.	2	3	4	5	6	7	8	9

The figures to the *left* of the decimal point represent *whole numbers;* those to the *right* are *decimals.*

In both the decimals and whole numbers, the *units* place is made the *starting point* of notation and numeration. Both whole numbers and decimals *decrease* on the scale of ten to the *right*, and both *increase* on the scale of ten to the *left*. The *first* figure to the *left* of units is *tens*, and the *first* figure to the *right* of units is *tenths*. The *second* figure to the *left* of units is *hundreds*, and the *second* figure to the *right* is *hundredths*. The *third* figure to the *left* is *thousands*, and the *third* to the *right* is *thousandths*, and so on; the *whole* numbers on the *left* and the *decimals* on the *right*. The figures equally distant from units place correspond in name, the *decimals* having the ending *ths*, to distinguish them from *whole* numbers. The following is the numeration of the number in the above table: nine hundred eighty-seven million, six hundred fifty-four thousand, three hundred twenty-one and twenty-three million, four hundred fifty-six thousand, seven hundred eighty-nine hundred-millionths.

The decimals increase to the *left*, on the scale of *ten*, the same as whole numbers; for, if you begin at the 4 in thousandths place in the above table, the next figure to the left is *hundredths*, which is ten times as great, and the next *tenths*, or ten times the *hundredths*, and so on through both decimals and whole numbers.

135. *Annexing, or taking away, a cipher at the right of a decimal, does not affect its value.*

.5 is $\frac{5}{10}$; .50 is $\frac{50}{100}$, but $\frac{5}{10} = \frac{50}{100}$; therefore, .5 = .50.

136. *Inserting a cipher between a decimal and the decimal point, divides the decimal by 10.*

$.5 = \frac{5}{10}$; $\frac{5}{10} \div 10 = \frac{5}{100} = .05$.

137. *Taking away a cipher from the left of a decimal, multiplies the decimal by 10.*

$.05 = \frac{5}{100}$; $\frac{5}{100} \times 10 = \frac{5}{10} = .5$.

138. In some cases it is convenient to express a mixed decimal fraction in the form of a common (improper) fraction. To do so it is only necessary to write the entire number, omitting the decimal point, as the numerator of the fraction, and the denominator of the decimal part as the denominator of the fraction. Thus, $127.483 = \frac{127483}{1000}$; for, $127.483 = 127\frac{483}{1000} = \frac{127000+483}{1000} = \frac{127483}{1000}$.

ADDITION OF DECIMALS.

139. Addition of decimals is similar in all respects to addition of whole numbers—units are placed under units, tens under tens, etc.; this, of course, brings the decimal points in line, directly under one another. Hence, in placing the numbers to be added, it is only necessary to take care that the *decimal points are in line.* In adding whole numbers, the right-hand figures are always in line; but in adding decimals, the right-hand figures will not be in line unless each decimal contains the same number of figures.

whole numbers	*decimals*	*mixed numbers*
342	.342	342.032
4234	.4234	4234.5
26	.26	26.6782
3	.03	3.06
sum 4605 Ans.	*sum* 1.0554 Ans.	*sum* 4606.2702 Ans.

140. EXAMPLE.—What is the sum of 242, .36, 118.725, 1.005, 6, and 100.1?

SOLUTION.—

242.
.36
118.725
1.005
6.
100.1
sum 468.190 Ans.

141. **Rule.**—*Place the numbers to be added so that the decimal points will be directly under each other. Add as in whole numbers, and place the decimal point in the sum directly under the decimal points above.*

EXAMPLES FOR PRACTICE.

142. Find the sum of:

(*a*) .2143, .105, 2.3042, and 1.1417.
(*b*) 783.5, 21.473, .2101, and .7816.
(*c*) 21.781, 138.72, 41.8738, .72, and 1.413.
(*d*) .3724, 104.15, 21.417, and 100.042.
(*e*) 200.172, 14.105, 12.1465, .705, and 7.2.
(*f*) 1,427.16, .244, .32, .032, and 10.0041.
(*g*) 2,473.1, 41.65, .7243, 104.067, and 21.073.
(*h*) 4,107.2, .00375, 21.716, 410.072, and .0345.

Ans.
(*a*) 3.7652.
(*b*) 805.9647.
(*c*) 204.5078.
(*d*) 225.9814.
(*e*) 234.3285.
(*f*) 1,437.7601.
(*g*) 2,640.6143.
(*h*) 4,539.02625.

SUBTRACTION OF DECIMALS.

143. As in subtraction of whole numbers, units are placed under units, tens under tens, etc., bringing the decimal points under each other, as in addition of decimals.

EXAMPLE.—Subtract .132 from .3063.

SOLUTION.—

minuend .3063
subtrahend .132
difference .1743 Ans.

144. EXAMPLE.—What is the difference between 7.895 and .725?

SOLUTION.—

minuend 7.895
subtrahend .725
difference 7.170 or 7.17. Ans.

145. EXAMPLE.—Subtract .625 from 11.

SOLUTION.—

minuend	11.000
subtrahend	.625
difference	10.375 Ans.

146. **Rule.**—*Place the subtrahend under the minuend, so that the decimal points will be directly under each other. Subtract as in whole numbers, and place the decimal point in the remainder directly under the decimal points above.*

When the figures in the decimal part of the subtrahend extend beyond those in the minuend, place ciphers in the minuend above them and subtract as before.

EXAMPLES FOR PRACTICE.

147. From:

(*a*)	407.385 take 235.0004.	(*a*)	172.3846.
(*b*)	22.718 take 1.7042.	(*b*)	21.0138.
(*c*)	1,368.17 take 13.6817.	(*c*)	1,354.4883.
(*d*)	70.00017 take 7.000017.	(*d*)	63.000153.
(*e*)	630.630 take .6304.	(*e*)	629.9996.
(*f*)	421.73 take 217.162.	(*f*)	204.568.
(*g*)	1.000014 take .00001.	(*g*)	1.000004.
(*h*)	.783652 take .542314.	(*h*)	.241338.

Ans.

MULTIPLICATION OF DECIMALS.

148. In multiplication of decimals we do not place the decimal points directly under each other as in addition and subtraction. We pay no attention for the time being to the decimal points. Place the multiplier under the multiplicand, so that the *right-hand* figure of the one is under the *right-hand* figure of the other, and proceed exactly as in multiplication of whole numbers. After multiplying, *count the number of decimal places in both multiplicand and multiplier, and point off the same number in the product.*

EXAMPLE.—Multiply .825 by 13.

SOLUTION.—

multiplicand	.825
multiplier	13
	2475
	825
product	10.725 Ans.

In this example there are 3 decimal places in the multiplicand and none in the multiplier; therefore, 3 decimal places are pointed off in the product.

149. EXAMPLE.—What is the product of 426 and the decimal .005?

SOLUTION.—

multiplicand	4 2 6
multiplier	.0 0 5
product	2.1 3 0 or 2.13. Ans.

In this example there are 3 decimal places in the multiplier and none in the multiplicand; therefore, 3 decimal places are pointed off in the product.

150. It is not necessary to multiply by the ciphers on the *left* of a decimal; they merely determine the *number of decimal places.* Ciphers to the *right* of a decimal should be omitted, as they only make more figures to deal with, and do not change the value.

151. EXAMPLE.—Multiply 1.205 by 1.15.

SOLUTION.—

multiplicand	1.2 0 5
multiplier	1.1 5
	6 0 2 5
	1 2 0 5
	1 2 0 5
product	1.3 8 5 7 5 Ans.

In this example there are 3 decimal places in the multiplicand and 2 in the multiplier; therefore, $3+2$, or 5, decimal places must be pointed off in the product.

152. EXAMPLE.—Multiply .232 by .001.

SOLUTION.—

multiplicand	.2 3 2
multiplier	.0 0 1
product	.0 0 0 2 3 2 Ans.

In this example we multiply the multiplicand by the digit in the multiplier, which gives 232 for the product; but since there are 3 decimal places each in the multiplier and multiplicand, we must prefix 3 ciphers to the 232 to make $3+3$, or 6, decimal places in the product.

153. **Rule.**—*Place the multiplier under the multiplicand, disregarding the position of the decimal points. Multiply*

as in whole numbers, and in the product point off as many decimal places as there are decimal places in both multiplier and multiplicand, prefixing ciphers if necessary.

EXAMPLES FOR PRACTICE.

154. Find the product of:

(*a*)	.000492 × 4.1418.	(*a*)	.0020377656.
(*b*)	4,003.2 × 1.2.	(*b*)	4,803.84.
(*c*)	78.6531 × 1.03.	(*c*)	81.012693.
(*d*)	.3685 × .042.	(*d*)	.015477.
(*e*)	178,352 × .01.	(*e*)	1,783.52.
(*f*)	.00045 × .0045.	(*f*)	.000002025.
(*g*)	.714 × .00002.	(*g*)	.00001428.
(*h*)	.00004 × .008.	(*h*)	.00000032.

Ans.

DIVISION OF DECIMALS.

155. In division of decimals we pay no attention to the decimal point until after the division has been performed. The *number of decimal places in the dividend must equal* (*or be made to equal by annexing ciphers*) *the number of decimal places in the divisor.* Divide exactly as in whole numbers. *Subtract the number of decimal places in the divisor from the number of decimal places in the dividend, and point off as many decimal places in the quotient as there are units in the remainder thus found.*

EXAMPLE.—Divide .625 by 25.

SOLUTION.—

divisor dividend quotient

25) .625 (.025 Ans.

50

125

125

remainder 0

In this example there are no decimal places in the divisor, and three decimal places in the dividend; therefore, there are 3 minus 0, or 3, decimal places in the quotient. One cipher has to be prefixed to the 25 to make the three decimal places.

156. EXAMPLE.—Divide 6.035 by .05.

divisor dividend quotient

SOLUTION.— .0 5) 6.0 3 5 (1 2 0.7 Ans.

5

1 0

1 0

3 5

3 5

remainder 0

In this example we divide by 5, as if the cipher were not before it. There is one more decimal place in the dividend than in the divisor; therefore, one decimal place is pointed off in the quotient.

157. EXAMPLE.—Divide .125 by .005.

divisor dividend quotient

SOLUTION.— .0 0 5) .1 2 5 (2 5 Ans.

1 0

2 5

2 5

remainder 0

In this example there are the same number of decimal places in the dividend as in the divisor; therefore, the quotient has no decimal places, and is a whole number.

158. EXAMPLE.—Divide 326 by .25.

divisor dividend quotient

SOLUTION.— .2 5) 3 2 6.0 0 (1 3 0 4 Ans.

2 5

7 6

7 5

1 0 0

1 0 0

remainder 0

In this problem two ciphers were annexed to the dividend, to make the number of decimal places equal to the number in the divisor. The quotient is a whole number.

159. EXAMPLE.—Divide .0025 by 1.25.

divisor dividend quotient

SOLUTION.— 1.2 5) .0 0 2 5 0 (.0 0 2 Ans.

2 5 0

remainder 0

EXPLANATION.—In this example we are to divide .0025 by 1.25. Consider the dividend as a whole number, i. e., as 25 (disregarding the two ciphers at its left, for the present); also, consider the divisor as a whole number, i. e., as 125. It is clearly evident that the dividend, 25, will not contain the divisor, 125; we must, therefore, annex one cipher to the 25, thus making the dividend 250. 125 is contained twice in 250, so we place the figure 2 in the quotient. In pointing off the decimal places in the quotient, it must be remembered that there were only four decimal places in the dividend; but one cipher was annexed, thereby making $4+1$, or 5, decimal places. Since there are five decimal places in the dividend and two decimal places in the divisor, we must point off $5-2$, or 3, decimal places in the quotient. In order to point off three decimal places, two ciphers must be prefixed to the figure 2, thereby making .002 the quotient. It is not necessary to consider the ciphers at the left of a decimal when dividing, except when determining the position of the decimal point in the quotient.

160. **Rule.—I.** *Place the divisor to the left of the dividend, and proceed as in division of whole numbers; in the quotient, point off as many decimal places as the number of decimal places in the dividend exceed those in the divisor, prefixing ciphers to the quotient, if necessary.*

II. *If in dividing one number by another there be a remainder, the remainder can be placed over the divisor, as a fractional part of the quotient, but it is generally better to annex ciphers to the remainder, and continue dividing until there are 3 or 4 decimal places in the quotient, and then if there still be a remainder, terminate the quotient by the plus sign* (+), *which shows that it can be carried further.*

161. EXAMPLE.—What is the quotient of 199 divided by 15?

divisor dividend quotient

SOLUTION.— $15)199(13+\frac{4}{15}$ Ans.

$$\begin{array}{r} 15 \\ \hline 49 \\ 45 \\ \hline \textit{remainder}\ \ 4 \end{array}$$

Or, $15)199.000(13.266+$ Ans.

$$\begin{array}{r} 15 \\ \hline 49 \\ 45 \\ \hline 40 \\ 30 \\ \hline 100 \\ 90 \\ \hline 100 \\ 90 \\ \hline \textit{remainder}\ \ 10 \end{array}$$

$13\frac{4}{15} = 13.266+$

$\frac{4}{15} = .266+$

162. It frequently happens, as in the above example, that the division will never terminate. In such cases, decide to how many decimal places the division is to be carried, and carry the work one place further. If the last figure of the quotient thus obtained is 5 or a greater number, increase the preceding figure by 1, and write after it the minus sign (−), thus indicating that the quotient is not quite as large as indicated; if the figure thus obtained is less than 5, write the plus sign (+) after the quotient, thus indicating that the number is slightly greater than as indicated. In the last example, had it been desired to obtain the answer correct to four decimal places, the work would have been carried to five places, obtaining 13.26666, and the answer would have been given as 13.2667−. This remark applies to any other calculation involving decimals, when it is desired to omit some of the figures in the decimal. Thus, if it was desired to retain three decimal places in the number .2471253, it would be expressed as .247+; if it was desired to retain five decimal places, it would be expressed as .24713−. Both the + and − signs are frequently omitted; they are

seldom used outside of arithmetic, except in exact calculations, when it is desired to call particular attention to the fact that the result obtained is not *quite* exact.

EXAMPLES FOR PRACTICE.

163. Divide:

(*a*)	101.6688 by 2.36.	Ans.	(*a*) 43.08.
(*b*)	187.12264 by 123.107.		(*b*) 1.52.
(*c*)	.08 by .008.		(*c*) 10.
(*d*)	.0003 by 3.75.		(*d*) .00008.
(*e*)	.0144 by .024.		(*e*) .6.
(*f*)	.00375 by 1.25.		(*f*) .003.
(*g*)	.004 by 400.		(*g*) .00001.
(*h*)	.4 by .008.		(*h*) 50.

REDUCTION OF DECIMALS.

TO REDUCE A FRACTION TO A DECIMAL.

164. EXAMPLE.— $\frac{3}{4}$ equals what decimal?

SOLUTION.—

$$4\overline{)3.00}\quad .75$$

or $\frac{3}{4} = .75$. Ans.

EXAMPLE.—What decimal is equivalent to $\frac{7}{8}$?

SOLUTION.—

$$8\overline{)7.000}\quad .875$$

64
60
56
40
40
0

or $\frac{7}{8} = .875$. Ans.

165. **Rule.**—*Annex ciphers to the numerator and divide by the denominator. Point off as many decimal places in the quotient as there are ciphers annexed.*

EXAMPLES FOR PRACTICE.

166. Reduce the following common fractions to decimals:

(*a*)	$\frac{15}{32}$.	Ans.	(*a*) .46875.
(*b*)	$\frac{7}{8}$.		(*b*) .875.
(*c*)	$\frac{21}{32}$.		(*c*) .65625.
(*d*)	$\frac{51}{64}$.		(*d*) .796875.
(*e*)	$\frac{4}{25}$.		(*e*) .16.
(*f*)	$\frac{5}{8}$.		(*f*) .625.
(*g*)	$\frac{10}{200}$.		(*g*) .05.
(*h*)	$\frac{4}{1000}$.		(*h*) .004.

167. To reduce inches to decimal parts of a foot:

EXAMPLE.—What decimal part of a foot is 9 inches?

SOLUTION.—Since there are 12 inches in one foot, 1 inch is $\frac{1}{12}$ of a foot, and 9 inches is $9 \times \frac{1}{12}$, or $\frac{9}{12}$ of a foot. This reduced to a decimal by the above rule shows what decimal part of a foot 9 inches is.

$$
\begin{array}{r}
12\,)\,9.00\,(.75 \text{ of a foot. Ans.} \\
\underline{84} \\
60 \\
\underline{60} \\
0
\end{array}
$$

168. Rule.—I. *To reduce inches to a decimal part of a foot, divide the number of inches by 12.*

II. *Should the resulting decimal be an unending one, and it is desired to terminate the division at some point, say the fourth decimal place, carry the division one place further, and if the fifth figure is 5 or greater increase the fourth figure by 1, omitting the signs + and −.*

EXAMPLES FOR PRACTICE.

169. Reduce to the decimal part of a foot:

(*a*)	3 in.		(*a*) .25 ft.
(*b*)	$4\frac{1}{2}$ in.		(*b*) .375 ft.
(*c*)	5 in.	Ans.	(*c*) .4167 ft.
(*d*)	$6\frac{5}{8}$ in.		(*d*) .5521 ft.
(*e*)	11 in.		(*e*) .9167 ft.

TO REDUCE A DECIMAL TO A FRACTION.

170. EXAMPLE.—Reduce .125 to a fraction.

SOLUTION.— $.125 = \frac{125}{1000} = \frac{5}{40} = \frac{1}{8}$. Ans.

EXAMPLE.—Reduce .875 to a fraction.

SOLUTION.— $.875 = \frac{875}{1000} = \frac{35}{40} = \frac{7}{8}$. Ans.

171. Rule.—*Under the figures of the decimal, place 1 with as many ciphers at its right as there are decimal places in the decimal, and reduce the resulting fraction to its lowest terms by dividing both numerator and denominator by the same number.*

EXAMPLES FOR PRACTICE.

172. Reduce the following to common fractions:

(*a*)	.125.	Ans.	(*a*) $\frac{1}{8}$.
(*b*)	.625.		(*b*) $\frac{5}{8}$.
(*c*)	.3125.		(*c*) $\frac{5}{16}$.
(*d*)	.04.		(*d*) $\frac{1}{25}$.
(*e*)	.06.		(*e*) $\frac{3}{50}$.
(*f*)	.75.		(*f*) $\frac{3}{4}$.
(*g*)	.15625.		(*g*) $\frac{5}{32}$.
(*h*)	.875.		(*h*) $\frac{7}{8}$.

173. To express a decimal approximatively as a fraction having a given denominator:

174. EXAMPLE.—Express .5827 in 64ths.

SOLUTION.— $.5827 \times \frac{64}{64} = \frac{37.2928}{64}$, say $\frac{37}{64}$.

Hence, .5827 = $\frac{37}{64}$, nearly. Ans.

EXAMPLE.—Express .3917 in 12ths.

SOLUTION.— $.3917 \times \frac{12}{12} = \frac{4.7004}{12}$, say $\frac{5}{12}$.

Hence, .3917 = $\frac{5}{12}$, nearly. Ans.

175. Rule.—*Reduce 1 to a fraction having the given denominator. Multiply the given decimal by the fraction so obtained, and the result will be the fraction required.*

EXAMPLES FOR PRACTICE.

176. Express:

(*a*)	.625 in 8ths.	Ans.	(*a*) $\frac{5}{8}$.
(*b*)	.3125 in 16ths.		(*b*) $\frac{5}{16}$.
(*c*)	.15625 in 32ds.		(*c*) $\frac{5}{32}$.
(*d*)	.77 in 64ths.		(*d*) $\frac{49}{64}$.
(*e*)	.81 in 48ths.		(*e*) $\frac{39}{48}$.
(*f*)	.923 in 96ths.		(*f*) $\frac{89}{96}$.

177. The sign for dollars is $. It is read *dollars.* $25 is read *25 dollars.*

Since there are 100 cents in a dollar, 1 cent is 1 one-hundredth of a dollar; the first two figures of a decimal part of

a dollar represent *cents*. Since a mill is $\frac{1}{10}$ of a cent, or $\frac{1}{1000}$ of a dollar, the third figure represents *mills*.

Thus, \$25.16 is read *twenty-five dollars and sixteen cents;* \$25.168 is read *twenty-five dollars sixteen cents and eight mills.*

SYMBOLS OF AGGREGATION.

178. The **vinculum** ——, **parenthesis** ( ), **brackets** [ ], and **brace** { } are called **symbols of aggregation,** and are used to include numbers which are to be considered together; thus, $13 \times \overline{8-3}$, or $13 \times (8-3)$, shows that 3 is to be taken from 8 before multiplying by 13.

$$13 \times (8-3) = 13 \times 5 = 65.$$

$$13 \times \overline{8-3} = 13 \times 5 = 65.$$

When the vinculum or parenthesis is not used, we have

$$13 \times 8 - 3 = 104 - 3 = 101.$$

179. In any series of numbers connected by the signs $+$, $-$, $\times$, and $\div$, the operations indicated by the signs must be performed in order from left to right, *except* that no addition or subtraction may be performed if a sign of multiplication or division *follows* the number on the *right* of a sign of addition or subtraction until the indicated multiplication or division has been performed. In all cases the sign of multiplication takes the precedence, the reason being that when two or more numbers or expressions are connected by the sign of multiplication the numbers thus connected are regarded as factors of the product indicated, and not as separate numbers.

EXAMPLE.—What is the value of $4 \times 24 - 8 + 17$?

SOLUTION.—Performing the operations in order from left to right, $4 \times 24 = 96$; $96 - 8 = 88$; $88 + 17 = 105$. Ans.

180. EXAMPLE.—What is the value of the following expression: $1{,}296 \div 12 + 160 - 22 \times 3\frac{1}{2} = ?$

SOLUTION.— $1{,}296 \div 12 = 108$; $108 + 160 = 268$; here we cannot subtract 22 from 268 because the sign of multiplication follows 22; hence, multiplying 22 by $3\frac{1}{2}$, we get 77, and $268 - 77 = 191$. Ans.

Had the above expression been written $1{,}296 \div 12 + 160 - 22 \times 3\frac{1}{2} \div 7 + 25$, it would have been necessary to have divided $22 \times 3\frac{1}{2}$ by 7 before subtracting, and the final result would have been $22 \times 3\frac{1}{2} = 77$; $77 \div 7 = 11$; $268 - 11 = 257$; $257 + 25 = 282$. Ans. In other words, it is necessary to perform all the indicated multiplication or division included between the signs + and −, or − and +, before adding or subtracting. Also, had the expression been written $1{,}296 \div 12 + 160 - 24\frac{1}{2} \div 7 \times 3\frac{1}{2} + 25$, it would have been necessary to have multiplied $3\frac{1}{2}$ by 7 before dividing $24\frac{1}{2}$, since the sign of multiplication takes the precedence, and the final result would have been $3\frac{1}{2} \times 7 = 24\frac{1}{2}$; $24\frac{1}{2} \div 24\frac{1}{2} = 1$; $268 - 1 = 267$; $267 + 25 = 292$. Ans.

It likewise follows that if a succession of multiplication and division signs occur, the indicated operations must not be performed in order, from left to right—the multiplication must be performed first. Thus, $24 \times 3 \div 4 \times 2 \div 9 \times 5 = \frac{1}{3}$. Ans. In order to obtain the same result that would be obtained by performing the indicated operations in order, from left to right, symbols of aggregation must be used. Thus, by using two vinculums the last expression becomes $24 \times \overline{3 \div 4} \times \overline{2 \div 9} \times 5 = 20$, the same result that would be obtained by performing the indicated operations in order, from left to right.

EXAMPLES FOR PRACTICE.

181. Find the values of the following expressions:

(*a*) $(8 + 5 - 1) \div 4$.
(*b*) $5 \times 24 - 32$.
(*c*) $5 \times 24 \div 15$.
(*d*) $144 - 5 \times 24$.
(*e*) $(1{,}691 - 540 + 559) \div 3 \times 57$.
(*f*) $2{,}080 \div 120 - 80 \times 4 - 1{,}670$.
(*g*) $(90 + 60 \div 25) \times 5 - 29$.
(*h*) $90 + 60 \div 25 \times 5$.

Ans. (*a*) 3. (*b*) 88. (*c*) 8. (*d*) 24. (*e*) 10. (*f*) 210. (*g*) 1. (*h*) 1.2.

ARITHMETIC.

(CONTINUED.)

PERCENTAGE.

1. **Percentage** is the process of calculating by *hundredths.*

2. The *term* **per cent.** is an abbreviation of the Latin words *per centum*, which mean *by the hundred.* A certain per cent. of a number is the number of hundredths of that number which is indicated by the number of units in the per cent. Thus, 6 per cent. of 125 is $125 \times \frac{6}{100} = 7.5$; 25 per cent. of 80 is $80 \times \frac{25}{100} = 20$; 43 per cent. of 432 pounds is $432 \times \frac{43}{100} = 185.76$ pounds.

3. The **sign** of per cent. is %, and is read *per cent.* Thus, 6% is read *six per cent.;* $12\frac{1}{2}\%$ is read *twelve and one-half per cent.*, etc.

When expressing the per cent. of a number to use in calculations, it is customary to express it decimally instead of fractionally. Thus, instead of expressing 6%, 25%, and 43% as $\frac{6}{100}$, $\frac{25}{100}$, and $\frac{43}{100}$, it is usual to express them as .06, .25, and .43.

The following table will show how any per cent. can be expressed either as a decimal or as a fraction:

Per Cent.	Decimal.	Fraction.	Per Cent.	Decimal.	Fraction.
1%.....	.01	$\frac{1}{100}$	150%.....	1.50	$\frac{150}{100}$ or $1\frac{1}{2}$
2%.....	.02	$\frac{2}{100}$ or $\frac{1}{50}$	500%.....	5.00	$\frac{500}{100}$ or 5
5%.....	.05	$\frac{5}{100}$ or $\frac{1}{20}$	$\frac{1}{4}\%$.....	.0025	$\frac{1}{100}$ or $\frac{1}{400}$
10%.....	.10	$\frac{10}{100}$ or $\frac{1}{10}$	$\frac{1}{2}\%$.....	.005	$\frac{1}{100}$ or $\frac{1}{200}$
25%.....	.25	$\frac{25}{100}$ or $\frac{1}{4}$	$1\frac{1}{2}\%$.....	.015	$\frac{1\frac{1}{2}}{100}$ or $\frac{3}{200}$
50%.....	.50	$\frac{50}{100}$ or $\frac{1}{2}$	$8\frac{1}{3}\%$.....	$.08\frac{1}{3}$	$\frac{8\frac{1}{3}}{100}$ or $\frac{1}{12}$
75%.....	.75	$\frac{75}{100}$ or $\frac{3}{4}$	$12\frac{1}{2}\%$.....	.125	$\frac{12\frac{1}{2}}{100}$ or $\frac{1}{8}$
100%.....	1.00	$\frac{100}{100}$ or 1	$16\frac{2}{3}\%$.....	$.16\frac{2}{3}$	$\frac{16\frac{2}{3}}{100}$ or $\frac{1}{6}$
125%.....	1.25	$\frac{125}{100}$ or $1\frac{1}{4}$	$62\frac{1}{2}\%$.....	.625	$\frac{62\frac{1}{2}}{100}$ or $\frac{5}{8}$

4. The names of the different elements used in percentage are: the *base*, the *rate per cent.*, the *percentage*, the *amount*, and the *difference*.

5. The **base** is the number on which the per cent. is computed.

6. The **rate** is the number of hundredths of the base to be taken.

7. The **percentage** is the part, or number of *hundredths*, of the base indicated by the rate; or, the percentage is the result obtained by multiplying the base by the rate.

Thus, when it is stated that 7% of \$25 is \$1.75, \$25 is the base, 7% is the rate, and \$1.75 is the percentage.

8. The **amount** is the sum of the base and percentage.

9. The **difference** is the remainder obtained by subtracting the percentage from the base.

Thus, if a man has \$180, and he earns 6% more, he will have altogether $\$180 + \$180 \times .06$, or $\$180 + \$10.80 = \$190.80$. Here \$180 is the base; 6%, the rate; \$10.80, the percentage; and \$190.80, the *amount*.

Again, if an engine of 125 horsepower uses 16% of it in overcoming friction and other resistances, the amount left for obtaining useful work is $125 - 125 \times .16 = 125 - 20 = 105$ horsepower. Here 125 is the base; 16%, the rate; 20, the percentage; and 105, the *difference*.

10. From the foregoing it is evident that to find the percentage, the base must be multiplied by the rate. Hence, the following

Rule.—*To find the percentage, multiply the base by the rate expressed decimally.*

EXAMPLE.—Out of a lot of 300 bushels of apples 76% were sold. How many bushels were sold?

SOLUTION.— 76%, the rate, expressed decimally, is .76; the base is 300; hence, the number of bushels sold, or the percentage, is, by the above rule,

$$300 \times .76 = 228 \text{ bushels.} \quad \text{Ans.}$$

Expressing the rule as a

Formula, $percentage = base \times rate$.

11. When the percentage and rate are given, the base may be found by dividing the percentage by the rate. For, suppose that 12 is 6%, or $\frac{6}{100}$, of some number; then 1%, or $\frac{1}{100}$, of the number, is $12 \div 6$, or 2. Consequently, if $2 = 1\%$, or $\frac{1}{100}$, 100%, or $\frac{100}{100} = 2 \times 100 = 200$. But, since the same result may be arrived at by dividing 12 by .06, for $12 \div .06 = 200$, it follows that.

Rule.—*When the percentage and rate are given, to find the base, divide the percentage by the rate expressed decimally.*

Formula, *base = percentage ÷ rate.*

EXAMPLE.—Bought a certain number of bushels of apples and sold 76% of them. If I sold 228 bushels, how many bushels did I buy?

SOLUTION.—Here 228 is the percentage, and 76%, or .76, is the rate; hence, applying the rule,

$228 \div .76 = 300$ bushels. Ans.

12. When the base and percentage are given, to find the rate, the rate may be found, expressed decimally, by dividing the percentage by the base. For, suppose that it is desired to find what per cent. 12 is of 200. 1% of 200 is $200 \times .01 = 2$. Now, if 1% is 2, 12 is evidently as many per cent. as the number of times that 2 is contained in 12, or $12 \div 2 = 6\%$. But the same result may be obtained by dividing 12, the percentage, by 200, the base, since $12 \div 200 = .06 = 6\%$. Hence,

Rule.—*When the percentage and base are given, to find the rate, divide the percentage by the base, and the result will be the rate expressed decimally.*

Formula, *rate = percentage ÷ base.*

EXAMPLE.—Bought 300 bushels of apples and sold 228 bushels. What per cent. of the total number of bushels was sold?

SOLUTION.—Here 300 is the base and 228 is the percentage; hence, applying rule,

rate $= 228 \div 300 = .76 = 76\%$. Ans.

EXAMPLE.—What per cent. of 875 is 25?

SOLUTION.—Here 875 is the base, and 25 is the percentage; hence, applying rule,

$25 \div 875 = .02\frac{4}{7} = 2\frac{4}{7}\%$. Ans.

PROOF.— $875 \times .02\frac{4}{7} = 25$.

EXAMPLES FOR PRACTICE.

13. What per cent. of:

(*a*)	360 is 90?	Ans.	(*a*) 25%.
(*b*)	900 is 360?		(*b*) 40%.
(*c*)	125 is 25?		(*c*) 20%.
(*d*)	150 is 750?		(*d*) 500%.
(*e*)	280 is 112?		(*e*) 40%.
(*f*)	400 is 200?		(*f*) 50%.
(*g*)	47 is 94?		(*g*) 200%.
(*h*)	500 is 250?		(*h*) 50%.

14. The amount may be found, when the base and rate are given, by multiplying the base by 1 plus the rate, expressed decimally. For, suppose that it is desired to find the amount when 200 is the base and 6% is the rate. The percentage is $200 \times .06 = 12$, and, according to definition, Art. **8**, the amount is $200 + 12 = 212$. But, the same result may be obtained by multiplying 200 by $1 + .06$, or 1.06, since $200 \times 1.06 = 212$. Hence,

Rule.—*When the base and rate are given, to find the amount, multiply the base by 1 plus the rate expressed decimally.*

Formula, *amount* = *base* $\times (1 +$ *rate*$)$.

EXAMPLE.—If a man earned $725 in a year, and the next year 10% more, how much did he earn the second year?

SOLUTION.—Here 725 is the base and 10% is the rate, and the amount is required. Hence, applying the rule,

$725 \times 1.10 = \$797.50$. Ans.

15. When the base and rate are given, the difference may be found by multiplying the base by 1 minus the rate expressed decimally. For, suppose that it is desired to find the difference when the base is 200 and the rate is 6%. The percentage is $200 \times .06 = 12$; and, according to definition, Art. **9**, the difference $= 200 - 12 = 188$. But, the same result may be obtained by multiplying 200 by $1 - .06$, or .94, since $200 \times .94 = 188$. Hence,

Rule.—*When the base and rate are given, to find the difference, multiply the base by 1 minus the rate expressed decimally.*

Formula, *difference* = *base* $\times (1 -$ *rate*$)$.

EXAMPLE.—Bought 300 bushels of apples and sold all but 24% of them. How many bushels were sold?

SOLUTION.—Here 300 is the base, 24% is the rate, and it is desired to find the difference. Hence, applying the rule,

$$300 \times (1 - .24) = 228 \text{ bushels.} \quad \text{Ans.}$$

16. When the amount and rate are given, the base may be found by dividing the amount by 1 plus the rate. For, suppose that it is known that 212 equals some number increased by 6% of itself. Then, it is evident that 212 equals 106% of the number (base) that it is desired to find. Consequently, if $212 = 106\%$, $1\% = \frac{212}{106} = 2$, and $100\% = 2 \times 100 = 200 =$ the base. But the same result may be obtained by dividing 212 by $1 + .06$, or 1.06, since $212 \div 1.06 = 200$. Hence,

Rule.—*When the amount and rate are given, to find the base, divide the amount by 1 plus the rate expressed decimally.*

Formula, *base* = *amount* $\div (1 +$ *rate*).

EXAMPLE.—The theoretical discharge of a certain pump when running at a piston speed of 100 feet per minute is 278,910 gallons per day of 10 hours. Owing to leakage and other defects, this value is 25% greater than the actual discharge. What is the actual discharge?

SOLUTION.—Here 278,910 equals the actual discharge (base) increased by 25% of itself. Consequently, 278,910 is the amount, and 25% is the rate. Applying rule,

actual discharge $= 278{,}910 \div 1.25 = 223{,}128$ gallons. Ans.

17. When the difference and rate are given, the base may be found by dividing the difference by 1 minus the rate. For, suppose that 188 equals some number less 6% of itself. Then, 188 evidently equals $100 - 6 = 94\%$ of some number. Consequently, if $188 = 94\%$, $1\% = 188 \div 94 = 2$, and $100\% = 2 \times 100 = 200$. But the same result may be obtained by dividing 188 by $1 - .06$, or .94, since $188 \div .94 = 200$. Hence,

Rule.—*When the difference and rate are given, to find the base, divide the difference by 1 minus the rate expressed decimally.*

Formula, *base* = *difference* $\div (1 -$ *rate*).

EXAMPLE.—Bought a certain number of bushels of apples and sold 76% of them. If there were 72 bushels left unsold, how many bushels did I buy?

SOLUTION.—Here 72 is the difference and 76% is the rate. Applying rule,

$$72 \div (1 - .76) = 300 \text{ bushels.} \quad \text{Ans.}$$

EXAMPLE.—The theoretical number of foot-pounds of work per minute required to operate a boiler feed-pump is 127,344. If 30% of the total number actually required be allowed for friction, leakage, etc., how many foot-pounds are actually required to work the pump?

SOLUTION.—Here the number actually required is the base; hence, 127,344 is the difference, and 30% is the rate. Applying the rule,

$$127{,}344 \div (1 - .30) = 181{,}920 \text{ foot-pounds.} \quad \text{Ans.}$$

18. EXAMPLE.—A certain chimney gives a draft of 2.76 inches of water. By increasing the height 20 feet, the draft was increased to 3 inches of water. What was the gain per cent.?

SOLUTION.—Here it is evident that 3 inches is the amount, and that 2.76 inches is the base. Consequently, $3 - 2.76 = .24$ inch is the percentage, and it is required to find the rate. Hence, applying the rule given in Art. **12**,

$$\text{gain per cent.} = .24 \div 2.76 = .087 = 8.7\%. \quad \text{Ans.}$$

19. EXAMPLE.—A certain chimney gave a draft of 3 inches of water. After an economizer had been put in, the draft was reduced to 1.2 inches of water. What was the loss per cent.?

SOLUTION.—Here it is evident that 1.2 inches is the difference (since it equals 3 inches diminished by a certain per cent. loss of itself), and 3 inches is the base. Consequently, $3 - 1.2 = 1.8$ inches is the percentage. Hence, applying the rule given in Art. **12**,

$$\text{loss per cent.} = 1.8 \div 3 = .60 = 60\%. \quad \text{Ans.}$$

20. To find the gain or loss per cent.:

Rule.—*Find the difference between the initial and the final value; divide this difference by the initial value.*

EXAMPLE.—If a man buys a house for $1,860, and some time afterwards builds a barn for 25% of the cost of the house, does he gain or lose, and how much per cent., if he sells both house and barn for $2,100?

SOLUTION.—The cost of the barn was $\$1{,}860 \times .25 = \$465$; consequently, the initial value, or total cost, was $\$1{,}860 + \$465 = \$2{,}325$. Since he sold them for $2,100 he lost $\$2{,}325 - \$2{,}100 = \$225$. Hence, applying rule,

$$225 \div 2{,}325 = .0968 = 9.68\% \text{ loss.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

21. Solve the following:

		Ans.	
(*a*)	What is $12\frac{1}{2}\%$ of \$900?	(*a*)	\$112.50.
(*b*)	What is $\frac{1}{4}\%$ of 627?	(*b*)	5.016.
(*c*)	What is $33\frac{1}{3}\%$ of 54?	(*c*)	18.
(*d*)	101 is $68\frac{3}{4}\%$ of what number?	(*d*)	$146\frac{10}{11}$.
(*e*)	784 is $83\frac{1}{3}\%$ of what number?	(*e*)	940.8.
(*f*)	What % of 960 is 160?	(*f*)	$16\frac{2}{3}\%$.
(*g*)	What % of \$3,606 is $\$450\frac{3}{4}$?	(*g*)	$12\frac{1}{2}\%$.
(*h*)	What % of 280 is 112?	(*h*)	40%.

1. A steam plant consumed an average of 3,640 pounds of coal per day. The engineer made certain alterations which resulted in a saving of 250 pounds per day. What was the per cent. of coal saved?

Ans. 7%, nearly.

2. If the speed of an engine running at 126 revolutions per minute should be increased $6\frac{1}{2}\%$, how many revolutions per minute would it then make? Ans. 134.19 rev.

3. The list price of a lot of silk goods is \$1,400, of some laces \$1,150, and of some calico \$340. If 25% discount was allowed on the silk, 22% on the laces, and $12\frac{1}{2}\%$ on the calico, what was the actual cost of the purchase? Ans. \$2,244.50.

4. If I loan a man \$1,100, and this is $18\frac{1}{2}\%$ of the amount that I have on interest, how much money have I on interest? Ans. \$5,945.95.

5. A test showed that an engine developed 190.4 horsepower, 15% of which was consumed in friction. How much power was available for use? Ans. 161.84 H. P.

6. By adding a condenser to a steam engine, the power was increased 14% and the consumption of coal per horsepower per hour was decreased 20%. If the engine could originally develop 50 horsepower, and required $3\frac{1}{2}$ pounds of coal per horsepower per hour, what would be the total weight of coal used in an hour, with the condenser, assuming the engine to run full power? Ans. 159.6 pounds.

DENOMINATE NUMBERS.

22. A **denominate number** is a concrete number, and may be either simple or compound; as, 8 quarts; 5 feet; 10 inches, etc.

23. A **simple denominate number** consists of units of but one denomination; as, 16 cents; 10 hours; 5 dollars, etc.

24. A **compound denominate number** consists of units of two or more denominations of a similar kind; as, 3 yards, 2 feet, 1 inch; 34 square feet, 57 square inches.

25. In **whole numbers** and in **decimals** the law of increase and decrease is on the scale of 10, but in **compound** or **denominate numbers** the scale varies.

26. A **measure** is a *standard unit*, established by *law* or *custom*, by which quantity of any kind is measured. The standard unit of **dry measure** is the Winchester bushel; of **weight,** the pound; of **liquid measure,** the gallon, etc.

27. Measures are of six kinds:

1. Extension.
2. Weight.
3. Capacity.
4. Time.
5. Angles.
6. Money or value.

MEASURES OF EXTENSION.

28. **Measures of extension** are used in measuring lengths, distances, surfaces, and solids.

LINEAR MEASURE.

TABLE.

Abbreviation.

12 inches (in.)	= 1 foot . .	ft.	in.		ft.		yd.		rd.	fur.	mi.
3 feet . . .	= 1 yard .	yd.	36	=	3	=	1				
5.5 yards . .	= 1 rod . .	rd.	198	=	$16\frac{1}{2}$	=	5.5	=	1		
40 rods . . .	= 1 furlong	fur.	7,920	=	660	=	220	=	40	= 1	
8 furlongs .	= 1 mile .	mi.	63,360	=	5,280	=	1,760	=	320	= 8	= 1

SURVEYOR'S LINEAR MEASURE.

TABLE.

7.92 inches	= 1 link	li.
25 links	= 1 rod	rd.
4 rods 100 links	= 1 chain	ch.
80 chains	= 1 mile	mi.

mi.		ch.		rd.		li.		in.
1	=	80	=	320	=	8,000	=	63,360

29. The linear unit, generally used by surveyors, is **Gunter's chain,** which is equal to 4 rods, or 66 feet.

30. An **engineer's chain,** used by civil engineers, is 100 feet long, and consists of 100 links. In computations, the links are written as so many hundredths of a chain.

SQUARE MEASURE.

TABLE.

144 square inches (sq. in.) . . . = 1 square foot . . . . sq. ft.
9 square feet . . . . . = 1 square yard . . . . sq. yd.
30¼ square yards . . . . . = 1 square rod . . . . sq. rd.
160 square rods . . . . . = 1 acre . . . . . A.
640 acres . . . . . = 1 square mile . . . . sq. mi.

sq. mi.	A.	sq. rd.	sq. yd.	sq. ft.	sq. in.
1	= 640	= 102,400	= 3,097,600	= 27,878,400	= 4,014,489,600

SURVEYOR'S SQUARE MEASURE.

TABLE.

625 square links (sq. li.) . . . . = 1 square rod . . . . sq. rd.
16 square rods . . . . . = 1 square chain . . . sq. ch.
10 square chains . . . . . = 1 acre . . . . . A.
640 acres. . . . . = 1 square mile . . . . sq. mi.
36 square miles (6 mi. square) . . = 1 township . . . . . Tp.

sq. mi.	A.	sq. ch.	sq. rd.	sq. li.
1	= 640	= 6,400	= 102,400	= 64,000,000

CUBIC MEASURE.

TABLE.

1,728 cubic inches (cu. in.) . . . = 1 cubic foot . . . . cu. ft.
27 cubic feet . . . . . = 1 cubic yard . . . . cu. yd.
128 cubic feet . . . . . = 1 cord . . . . . cd.
24¾ cubic feet . . . . . = 1 perch . . . . . P.

cu. yd.	cu. ft.	cu. in.
1	= 27	= 46,656

MEASURES OF WEIGHT.

AVOIRDUPOIS WEIGHT.

TABLE.

16 ounces (oz.) . . . . . = 1 pound . . . . . lb.
100 pounds . . . . . = 1 hundredweight . . . cwt.
20 cwt., or 2,000 lb. . . . . = 1 ton . . . . . T.

T.	cwt.	lb.	oz.
1	= 20	= 2,000	= 32,000

31. The ounce is divided into halves, quarters, etc. Avoirdupois weight is used for weighing coarse and heavy articles. One avoirdupois pound contains 7,000 grains.

LONG TON TABLE.

16 ounces	=	1 pound	lb.
112 pounds	=	1 hundredweight	cwt.
20 cwt., or 2,240 lb.	=	1 ton	T.

32. In all the calculations throughout this and the following sections, 2,000 pounds will be considered 1 ton, unless the long ton (2,240 pounds) is especially mentioned.

TROY WEIGHT.

TABLE.

24 grains (gr.)	=	1 pennyweight	pwt.
20 pennyweights	=	1 ounce	oz.
12 ounces	=	1 pound	lb.

lb.	oz.	pwt.	gr.
1	= 12	= 240	= 5,760

33. Troy weight is used in weighing gold and silverware, jewels, etc. It is used by jewelers.

MEASURES OF CAPACITY.

LIQUID MEASURE.

TABLE.

4 gills (gi.)	=	1 pint	pt.
2 pints	=	1 quart	qt.
4 quarts	=	1 gallon	gal.
31½ gallons	=	1 barrel	bbl.
2 barrels, or 63 gallons	=	1 hogshead	hhd.

hhd.	bbl.	gal.	qt.	pt.	gi.
1 =	2 =	63 =	252 =	504 =	2,016

DRY MEASURE.

TABLE.

2 pints (pt.)	=	1 quart	qt.
8 quarts	=	1 peck	pk.
4 pecks	=	1 bushel	bu.

bu.	pk.	qt.	pt.
1 =	4 =	32 =	64

MEASURE OF TIME.

TABLE.

60 seconds (sec.) . . . . . = 1 minute . . . . . min.
60 minutes . . . . . = 1 hour . . . . . hr.
24 hours . . . . . = 1 day . . . . . da.
7 days . . . . . = 1 week . . . . . wk.
365 days / 12 months . . . . . = 1 common year . . . . yr.
366 days . . . . . = 1 leap year.
100 years . . . . . = 1 century.

NOTE.—It is customary to consider one month as 30 days.

MEASURE OF ANGLES OR ARCS.

TABLE.

60 seconds (″) . . . . . = 1 minute . . . . . ′.
60 minutes . . . . . = 1 degree . . . . . °.
90 degrees . . . . . = 1 right angle or quadrant ⌊.
360 degrees . . . . . = 1 circle . . . . . cir.

1 cir. = 360° = 21,600′ = 1,296,000″

MEASURE OF MONEY.

UNITED STATES MONEY.

TABLE.

10 mills (m.) . . . . . = 1 cent . . . . . ct.
10 cents . . . . . = 1 dime . . . . . d.
10 dimes . . . . . = 1 dollar . . . . . $.
10 dollars . . . . . = 1 eagle . . . . . E.

E. $ d. ct. m.
1 = 10 = 100 = 1,000 = 10,000

MISCELLANEOUS TABLE.

12 things are 1 dozen.
12 dozen are 1 gross.
12 gross are 1 great gross.
2 things are 1 pair.
20 things are 1 score.
1 league is 3 miles.
1 fathom is 6 feet.
1 meter is nearly 39.37 inches.
1 hand is 4 inches.
1 palm is 3 inches.
1 span is 9 inches.
24 sheets are 1 quire.
20 quires, or 480 sheets, are 1 ream.
1 bushel contains 2,150.4 cubic in.

1 U. S. standard gallon (also called a wine gallon) contains 231 cubic in.
1 U. S. standard gallon of water weighs 8.355 pounds, nearly.
1 cubic foot of water contains 7.481 U. S. standard gallons, nearly.
1 British imperial gallon weighs 10 pounds.

It will be of great advantage to the student to carefully memorize all the above tables.

REDUCTION OF DENOMINATE NUMBERS.

34. **Reduction** of denominate numbers is the process of changing their denomination without changing their value. They may be changed from a higher to a lower denomination, or from a lower to a higher—either is reduction. As

2 hours = 120 minutes.
32 ounces = 2 pounds.

35. **Principle.**—Denominate numbers are changed to lower denominations by *multiplying*, and to higher denominations by *dividing*.

To reduce denominate numbers to lower denominations:

36. EXAMPLE.—Reduce 5 yd. 2 ft. 7 in. to inches.

SOLUTION.—

yd.	ft.	in.
5	2	7
3		
15 ft.		
2 ft.		
17 ft.		
12		
34		
17		
204 in.		
7 in.		
211 inches. Ans.		

EXPLANATION.—Since there are 3 feet in 1 yard, in 5 yards there are 5×3 or 15 feet, and 15 feet plus 2 feet = 17 feet. There are 12 inches in a foot; therefore, $12 \times 17 = 204$ inches, and 204 inches plus 7 inches = 211 inches = number of inches in 5 yards 2 feet and 7 inches.

37. EXAMPLE.—Reduce 6 hours to seconds.

SOLUTION.—

6	hours.
60	
360	minutes.
60	
21600	seconds. Ans.

EXPLANATION.—As there are 60 minutes in 1 hour, in 6 hours there are 6×60, or 360, minutes; as there are no minutes to add, we multiply 360 minutes by 60, to get the number of seconds.

38. In order to avoid mistakes, if any denomination be omitted, represent it by a cipher. Thus, before reducing 3 rods 6 inches to inches, insert a cipher for yards and a cipher for feet, as

rd.	yd.	ft.	in.
3	0	0	6

39. Rule.—*Multiply the number representing the highest denomination by the number of units in the next lower required to make one of the higher denomination, and to the product add the number of given units of that lower denomination. Proceed in this manner until the number is reduced to the required denomination.*

EXAMPLES FOR PRACTICE.

40. Reduce:

(*a*) 4 rd. 2 yd. 2 ft. to ft.
(*b*) 4 bu. 3 pk. 2 qt. to qt.
(*c*) 13 rd. 5 yd. 2 ft. to ft.
(*d*) 5 mi. 100 rd. 10 ft. to ft.
(*e*) 8 lb. 4 oz. 6 pwt. to gr.
(*f*) 52 hhd. 24 gal. 1 pt. to pt.
(*g*) 5 cir. 16° 20′ to minutes.
(*h*) 14 bu. to qt.

Ans. (*a*) 74 ft. (*b*) 154 qt. (*c*) 231.5 ft. (*d*) 28,060 ft. (*e*) 48,144 gr. (*f*) 26,401 pt. (*g*) 108,980′. (*h*) 448 qt.

To reduce lower to higher denominations:

41. EXAMPLE.—Reduce 211 inches to higher denominations.

SOLUTION.—

$$\begin{array}{r} 12\,)\,2\,1\,1 \text{ in.} \\ \hline 3\,)\,1\,7 \text{ ft.} + 7 \text{ in.} \\ \hline 5 \text{ yd.} + 2 \text{ ft.} \end{array}$$ Ans.

EXPLANATION.—There are 12 inches in 1 foot; therefore, 211 divided by 12 = 17 feet and 7 inches over. There are 3 feet in 1 yard; therefore, 17 feet divided by 3 = 5 yards

and 2 feet over. The last quotient and the two remainders constitute the answer, 5 yards 2 feet 7 inches.

42. EXAMPLE.—Reduce 15,735 grains Troy weight to higher denominations.

SOLUTION.—

24) 15735 gr. (655 pwt.
144
133
120
135
120
15 gr.

20) 655 pwt. (32 oz.
60
55
40
15 pwt.

12) 32 oz. (2 lb.
24
8 oz.

EXPLANATION.—There are 24 grains in 1 pennyweight, and in 15,735 grains there are as many pennyweights as 24 is contained in 15,735, or 655 pennyweights and 15 grains remaining. There are 20 pennyweights in 1 ounce, and in 655 pennyweights there are 32 ounces and 15 pennyweights remaining. There are 12 ounces in 1 pound, and in 32 ounces there are 2 pounds and 8 ounces remaining. The last quotient and the three remainders constitute the answer, 2 pounds 8 ounces 15 pennyweights 15 grains.

The above problem is worked out by long division, because the numbers are too large to solve easily by short division. The student may use either method.

43. **Rule.**—*Divide the number representing the denomination given by the number of units of this denomination required to make one unit of the next higher denomination. The remainder will be of the same denomination, but the quotient will be of the next higher. Divide this quotient by the number of units of its denomination required to make one unit of the next higher. Continue until the highest*

denomination is reached, or until there is not enough of a denomination left to make one of the next higher. The last quotient and the remainders constitute the required result.

EXAMPLES FOR PRACTICE.

44. Reduce to units of higher denominations:

(*a*) 7,460 sq. in.; (*b*) 7,580 sq. yd.; (*c*) 148,760 cu. in.; (*d*) 7.896 cu. ft. to cd.; (*e*) 17,651″; (*f*) 1,120 cu. ft. to cd.; (*g*) 8,000 gi.; (*h*) 36,450 lb.

Ans.
(*a*) 5 sq. yd. 6 sq. ft. 116 sq. in.
(*b*) 1 A. 90 sq. rd. 17 sq. yd. 4 sq. ft. 72 sq. in.
(*c*) 3 cu. yd. 5 cu. ft. 152 cu. in.
(*d*) 61 cd. 88 cu. ft.
(*e*) 4° 54′ 11″.
(*f*) 8 cd. 96 cu. ft.
(*g*) 3 hhd. 61 gal.
(*h*) 18 T. 4 cwt. 50 lb.

ADDITION OF DENOMINATE NUMBERS.

45. EXAMPLE.—Find the sum of 3 cwt. 46 lb. 12 oz.; 8 cwt. 12 lb. 13 oz.; 12 cwt. 50 lb. 13 oz.; 27 lb. 4 oz.

SOLUTION.—

T.	cwt.	lb.	oz.	
0	3	46	12	
0	8	12	13	
0	12	50	13	
0	0	27	4	
1	4	37	10	Ans.

EXPLANATION.—Begin to add at the right-hand column: $4+13+13+12 = 42$ ounces; as 16 ounces make 1 pound, 42 ounces $\div 16 = 2$ and a remainder of 10 ounces, or 2 pounds and 10 ounces. Place 10 ounces under ounce column and add 2 pounds to the next or pound column. Then, $2+27+50+12+46 = 137$ pounds; as 100 pounds make a hundredweight, $137 \div 100 = 1$ hundredweight and a remainder of 37 pounds. Place the 37 under the pounds column, and add 1 hundredweight to the next or hundredweight column. Next, $1+12+8+3 = 24$ hundredweight.

20 hundredweight make a ton; therefore $24 \div 20 = 1$ ton and 4 hundredweight remaining. Hence, the sum is 1 ton 4 hundredweight 37 pounds 10 ounces. Ans.

46. EXAMPLE.—What is the sum of 2 rd. 3 yd. 2 ft. 5 in.; 6 rd. 1 ft. 10 in.; 17 rd. 11 in.; 4 yd. 1 ft.?

SOLUTION.—

	rd.	yd.	ft.	in.	
	2	3	2	5	
	6	0	1	10	
	17	0	0	11	
	0	4	1	0	
	26	$3\frac{1}{2}$	0	2	
or	26	3	1	8	Ans.

EXPLANATION.—The sum of the numbers in the first column = 26 inches, or 2 feet and 2 inches remaining. The sum of the numbers in the next column plus 2 feet = 6 feet, or 2 yards and 0 feet remaining. The sum of the next column plus 2 yards = 9 yards, or $9 \div 5\frac{1}{2} = 1$ rod and $3\frac{1}{2}$ yards remaining. The sum of the next column plus 1 rod = 26 rods. To avoid fractions in the sum, the $\frac{1}{2}$ yard is reduced to 1 foot and 6 inches, which added to 26 rods 3 yards 0 feet and 2 inches = 26 rods 3 yards 1 foot 8 inches. Ans.

47. EXAMPLE.—What is the sum of 47 ft. and 3 rd. 2 yd. 2 ft. 10 in.?

SOLUTION.—When 47 ft. is reduced it equals 2 rd. 4 yd. 2 ft. which can be added to 3 rd. 2 yd. 2 ft. 10 in. Thus,

	rd.	yd.	ft.	in.	
	3	2	2	10	
	2	4	2	0	
	6	$1\frac{1}{2}$	1	10	
or	6	2	0	4	Ans.

48. Rule.—*Place the numbers so that like denominations are under each other. Begin at the right-hand column, and add. Divide the sum by the number of units of this denomination required to make one unit of the next higher. Place the remainder under the column added, and carry the quotient to the next column. Continue in this manner until the highest denomination given is reached.*

EXAMPLES FOR PRACTICE.

49. What is the sum of:

(*a*) 25 lb. 7 oz. 15 pwt. 23 gr.; 17 lb. 16 pwt.; 15 lb. 4 oz. 12 pwt.; 18 lb. 16 gr.; 10 lb. 2 oz. 11 pwt. 16 gr.?

(*b*) 9 mi. 13 rd. 4 yd. 2 ft.; 16 rd. 5 yd. 1 ft. 5 in.; 16 mi. 2 rd. 3 in.; 14 rd. 1 yd. 9 in.?

(*c*) 3 cwt. 46 lb. 12 oz.; 12 cwt. $9\frac{1}{2}$ lb.; $2\frac{1}{4}$ cwt. $21\frac{5}{8}$ lb.?

(*d*) 10 yr. 8 mo. 5 wk. 3 da.; 42 yr. 6 mo. 7 da.; 7 yr. 5 mo. 18 wk. 4 da.; 17 yr. 17 da.?

(*e*) 17 T. 11 cwt. 49 lb. 14 oz.; 16 T. 47 lb. 13 oz.; 20 T. 13 cwt. 14 lb. 6 oz.; 11 T. 4 cwt. 16 lb. 12 oz.?

(*f*) 14 sq. yd. 8 sq. ft. 19 sq. in.; 105 sq. yd. 16 sq. ft. 240 sq. in.; 42 sq. yd. 28 sq. ft. 165 sq. in.?

Ans. (*a*) 86 lb. 3 oz. 16 pwt. 7 gr.
(*b*) 25 mi. 47 rd. 1 ft. 5 in.
(*c*) 18 cwt. 2 lb. 14 oz.
(*d*) 78 yr. 1 mo. 3 wk. 3 da.
(*e*) 65 T. 9 cwt. 28 lb. 13 oz.
(*f*) 167 sq. yd. 136 sq. in.

SUBTRACTION OF DENOMINATE NUMBERS.

50. EXAMPLE.—From 21 rd. 2 yd. 2 ft. $6\frac{1}{2}$ in. take 9 rd. 4 yd. $10\frac{1}{4}$ in.

SOLUTION.—

rd.	yd.	ft.	in.	
21	2	2	$6\frac{1}{2}$	
9	4	0	$10\frac{1}{4}$	
11	$3\frac{1}{2}$	1	$8\frac{1}{4}$	Ans.

EXPLANATION.—Since $10\frac{1}{4}$ inches cannot be taken from $6\frac{1}{2}$ inches, we must borrow 1 foot or 12 inches from the 2 feet in the next column and add it to the $6\frac{1}{2}$. $6\frac{1}{2}+12=18\frac{1}{2}$. $18\frac{1}{2}$ inches $-10\frac{1}{4}$ inches $=8\frac{1}{4}$ inches. Then, 0 from the 1 remaining foot $=1$ foot. 4 yards cannot be taken from 2 yards; therefore, we borrow 1 rod, or $5\frac{1}{2}$ yards, from 21 rods and add it to 2. $2+5\frac{1}{2}=7\frac{1}{2}$; $7\frac{1}{2}-4=3\frac{1}{2}$ yards. 9 rods from 20 rods $=11$ rods. Hence, the remainder is 11 rods $3\frac{1}{2}$ yards 1 foot $8\frac{1}{4}$ inches. Ans.

To avoid fractions as much as possible, we reduce the $\frac{1}{2}$ yard to inches, obtaining 18 inches; this added to $8\frac{1}{4}$ inches gives $26\frac{1}{4}$ inches, which equals 2 feet $2\frac{1}{4}$ inches. Then, 2 feet $+1$ foot $=3$ feet $=1$ yard, and 3 yards $+1$ yard $=4$

yards. Hence, the above answer becomes 11 rods 4 yards 0 feet $2\frac{1}{4}$ inches.

51. EXAMPLE.—What is the difference between 3 rd. 2 yd. 2 ft. 10 in. and 47 ft. ?

SOLUTION.— 47 ft. = 2 rd. 4 yd. 2 ft.

	rd.	yd.	ft.	in.	
	3	2	2	10	
	2	4	2	0	
	0	$3\frac{1}{2}$	0	10	
or		3	2	4	Ans.

To find (approximately) the interval of time between two dates:

52. EXAMPLE.—How many years, months, days, and hours between 4 o'clock P. M. of June 16, 1868, and 10 o'clock A. M., September 29, 1891?

SOLUTION.—

yr.	mo.	da.	hr.	
1891	8	28	10	
1868	5	15	16	
23	3	12	18	Ans.

EXPLANATION.—Counting 24 hours in 1 day, 4 o'clock P. M. is the 16th hour from the beginning of the day, or midnight. On September 29, 8 months and 28 days have elapsed, and on June 16, 5 months and 15 days. After placing the earlier date under the later date, subtract as in the previous problems. Count 30 days as 1 month.

53. **Rule.**—*Place the smaller quantity under the larger quantity, with like denominations under each other. Beginning at the right, subtract successively the number in the subtrahend in each denomination from the one above, and place the differences underneath. If the number in the minuend of any denomination is less than the number under it in the subtrahend, one must be borrowed from the minuend of the next higher denomination, reduced, and added to it.*

EXAMPLES FOR PRACTICE.

54. From:

(*a*) 125 lb. 8 oz. 14 pwt. 18 gr. take 96 lb. 9 oz. 10 pwt. 4 gr.

(*b*) 126 hhd. 27 gal. take 104 hhd. 14 gal. 1 qt. 1 pt.

(*c*) 65 T. 14 cwt. 64 lb. 10 oz. take 16 T. 11 cwt. 14 oz.

(*d*) 148 sq. yd. 16 sq. ft. 142 sq. in. take 132 sq. yd. 136 sq. in.

(*e*) 100 bu. take 28 bu. 2 pk. 5 qt. 1 pt.

(*f*) 14 mi. 34 rd. 16 yd. 13 ft. 11 in. take 3 mi. 27 rd. 11 yd. 4 ft. 10 in.

Ans. (*a*) 28 lb. 11 oz. 4 pwt. 14 gr.
(*b*) 22 hhd. 12 gal. 2 qt. 1 pt.
(*c*) 49 T. 3 cwt. 63 lb. 12 oz.
(*d*) 16 sq. yd. 16 sq. ft. 6 sq. in.
(*e*) 71 bu. 1 pk. 2 qt. 1 pt.
(*f*) 11 mi. 7 rd. 5 yd. 9 ft. 1 in.

MULTIPLICATION OF DENOMINATE NUMBERS.

55. EXAMPLE.—Multiply 7 lb. 5 oz. 13 pwt. 15 gr. by 12.

SOLUTION.—

lb.	oz.	pwt.	gr.	
7	5	13	15	
			12	
89	8	3	12	Ans.

EXPLANATION.— 15 grains $\times 12 = 180$ grains. $180 \div 24 = 7$ pennyweights and 12 grains remaining. Place the 12 in the grain column and carry the 7 pennyweights to the next. Now, $13 \times 12 + 7 = 163$ pennyweights; $163 \div 20 = 8$ ounces and 3 pennyweights remaining. Then, $5 \times 12 + 8 = 68$ ounces; $68 \div 12 = 5$ pounds and 8 ounces remaining. Then, $7 \times 12 + 5 = 89$ pounds. The entire product is 89 pounds 8 ounces 3 pennyweights 12 grains. Ans.

56. **Rule.**—*Multiply the number representing each denomination by the multiplier and reduce each product to the next higher denomination, writing the remainders under each denomination, and carry the quotient to the next, as in Addition of Denominate Numbers.*

57. In multiplication and division of denominate numbers, it is sometimes easier to reduce the number to the lowest denomination given before multiplying or dividing, especially if the multiplier or divisor is a decimal. Thus, in the example of Art. **55**, had the multiplier been 1.2, the easiest way to multiply would have been to reduce the number to grains; then, multiply by 1.2, and reduce the product to higher denominations. For example, 7 lb. 5 oz. 13 pwt. 15 gr. = 43,047 gr. $43{,}047 \times 1.2 = 51{,}656.4$ gr. = 8 lb. 11 oz. 12 pwt. 8.4 gr. Also, $43{,}047 \times 12 = 516{,}564$ gr. = 89 lb. 8 oz. 3 pwt. 12 gr., as above. Either method may be used.

EXAMPLES FOR PRACTICE.

58. Multiply:

(*a*) 15 cwt. 90 lb. by 5; (*b*) 12 yr. 10 mo. 4 wk. 3 da. by 14; (*c*) 11 mi. 145 rd. by 20; (*d*) 12 gal. 4 pt. by 9; (*e*) 8 cd. 76 cu. ft. by 15; (*f*) 4 hhd. 3 gal. 1 qt. 1 pt. by 12.

Ans. (*a*) 79 cwt. 50 lb.
(*b*) 180 yr. 11 mo. 2 wk.
(*c*) 229 mi. 20 rd.
(*d*) 112 gal. 2 qt.
(*e*) 128 cd. 116 cu. ft.
(*f*) 48 hhd. 40 gal. 2 qt.

DIVISION OF DENOMINATE NUMBERS.

59. EXAMPLE.—Divide 48 lb. 11 oz. 6 pwt. by 8.

SOLUTION.—

	lb.	oz.	pwt.	gr.
8)	48	11	6	0
	6 lb.	1 oz.	8 pwt.	6 gr. Ans.

EXPLANATION.—After placing the quantities as above, proceed as follows: 8 is contained in 48 six times without a remainder. 8 is contained in 11 ounces once, with 3 ounces remaining. $3 \times 20 = 60$; $60 + 6 = 66$ pennyweights; 66 pennyweights $\div 8 = 8$ pennyweights and 2 remaining; 2×24 grains $= 48$ grains; 48 grains $\div 8 = 6$ grains. Therefore, the entire quotient is 6 pounds 1 ounce 8 pennyweights 6 grains. Ans.

EXAMPLE.—A silversmith melted up 2 lb. 8 oz. 10 pwt. of silver, which he made into 6 spoons; what was the weight of each spoon?

SOLUTION.—

	lb.	oz.	pwt.	
6)	2	8	10	
		5 oz.	8 pwt.	8 gr. Ans.

EXPLANATION.—Since we cannot divide 2 pounds by 6, we reduce it to ounces. 2 pounds $= 24$ ounces, and 24 ounces $+ 8$ ounces $= 32$ ounces; 32 ounces $\div 6 = 5$ ounces and 2 ounces over. 2 ounces $= 40$ pennyweights; 40 pennyweights $+ 10$ pennyweights $= 50$ pennyweights, and 50 pennyweights $\div 6 = 8$ pennyweights and 2 pennyweights over. 2 pennyweights $= 48$ grains, and 48 grains $\div 6 = 8$ grains. Hence, each spoon contains 5 ounces 8 pennyweights 8 grains. Ans.

60. EXAMPLE.—Divide 820 rd. 4 yd. 2 ft. by 112.

SOLUTION.—

```
                    rd.  yd. ft.  rd. yd. ft.  in.
112)820  4  2 (  7  1  2  5.143 Ans.
    784
     36 rd. rem.
      5.5
     180
    180
    198.0 yd.
      4
112)202 yd. ( 1 yd.
    112
     90 yd. rem.
      3
     270 ft.
       2 ft.
112)272 ft. ( 2 ft.
    224
     48 ft. rem.
     12
     96
    48
112)576 in. ( 5.1428+ in. or 5.143 in.
    560
     160
     112
      480
      448
       320
       224
        960
        896
         64
```

EXPLANATION.—The first quotient is 7 rods with 36 rods remaining. $5.5 \times 36 = 198$ yards; 198 yards + 4 yards = 202 yards; 202 yards $\div 112 = 1$ yard and 90 yards remaining. $90 \times 3 = 270$ feet; 270 feet + 2 feet = 272 feet; 272 feet $\div 112 = 2$ feet, and 48 feet remaining; $48 \times 12 = 576$ inches; 576 inches $\div 112 = 5.143$ inches, nearly. Ans.

The preceding example is solved by long division, because

the numbers are too large to deal with mentally. Instead of expressing the last result as a decimal, it might have been expressed as a common fraction. Thus, $576 \div 112 = 5\frac{16}{112} = 5\frac{1}{7}$ inches. The chief advantage of using a common fraction is that if the quotient be multiplied by the divisor, the result will always be the same as the original dividend.

61. Rule.—*Find how many times the divisor is contained in the first or highest denomination of the dividend. Reduce the remainder (if any) to the next lower denomination, and add to it the number in the given dividend expressing that denomination. Divide this new dividend by the divisor. The quotient will be the next denomination in the quotient required. Continue in this manner until the lowest denomination is reached. The successive quotients will constitute the entire quotient.*

EXAMPLES FOR PRACTICE.

62. Divide:

(*a*) 376 mi. 276 rd. by 22; (*b*) 1,137 bu. 3 pk. 4 qt. 1 pt. by 10; (*c*) 84 cwt. 48 lb. 49 oz. by 16; (*d*) 78 sq. yd. 18 sq. ft. 41 sq. in. by 18; (*e*) 148 mi. 64 rd. 24 yd. by 12; (*f*) 100 T. 16 cwt. 18 lb. 11 oz. by 15; (*g*) 36 lb. 18 oz. 18 pwt. 14 gr. by 8; (*h*) 112 mi. 48 rd. by 100.

Ans.
(*a*) 17 mi. $41\frac{7}{11}$ rd.
(*b*) 113 bu. 3 pk. 1 qt. $\frac{1}{2}$ pt.
(*c*) 5 cwt. 28 lb. $3\frac{1}{16}$ oz.
(*d*) 4 sq. yd. 4 sq. ft. $2\frac{5}{18}$ sq. in.
(*e*) 12 mi. 112 rd. 2 yd.
(*f*) 6 T. 14 cwt. 41 lb. $3\frac{14}{15}$ oz.
(*g*) 4 lb. 8 oz. 7 pwt. $7\frac{3}{4}$ gr.
(*h*) 1 mi. $38\frac{88}{225}$ rd.

INVOLUTION.

63. Involution is the process of multiplying a number by itself one or more times. The product obtained by multiplying a number by itself is called a **power** of that number.

Thus, the *second* power of 3 is 9, since 3×3 are 9.

The *third* power of 3 is 27, since $3 \times 3 \times 3$ are 27.

The *fifth* power of 2 is 32, since $2 \times 2 \times 2 \times 2 \times 2$ are 32.

64. An **exponent** is a *small figure* placed to the *right* and a little *above* a number to show to what *power* it is to be raised, or how many times the number is to be used as a factor, as the small figures $^{2,\ 3,}$ and ^{5} below.

Thus,
$$3^2 = 3 \times 3 = 9.$$
$$3^3 = 3 \times 3 \times 3 = 27.$$
$$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32.$$

65. The **root** of a number is that number which, used the required number of times as a factor, produces the number. In the above cases 3 is a root of 9, since 3×3 are 9. It is also a root of 27, since $3 \times 3 \times 3$ are 27. Also, 2 is a root of 32, since $2 \times 2 \times 2 \times 2 \times 2$ are 32.

66. The *second* power of a number is called its **square.**

Thus, 5^2 is called the *square* of 5, or 5 *squared*, and its value is $5 \times 5 = 25$.

67. The *third* power of a number is called its **cube.**

Thus, 5^3 is called the *cube* of 5, or 5 *cubed*, and its value is $5 \times 5 \times 5 = 125$.

To find any power of a number:

68. Example.—What is the third power, or cube, of 35?

Solution.—

$$\begin{array}{rr}
 & 35 \times 35 \times 35, \\
\text{or} & 35 \\
 & 35 \\
\hline
 & 175 \\
 & 105 \\
\hline
 & 1225 \\
 & 35 \\
\hline
 & 6125 \\
 & 3675 \\
\hline
\textit{cube} = & 42875 \quad \text{Ans.}
\end{array}$$

EXAMPLE.—What is the fourth power of 15?

SOLUTION.— $15 \times 15 \times 15 \times 15$,

or

$$\begin{array}{r} 15 \\ 15 \\ \hline 75 \\ 15 \\ \hline 225 \\ 15 \\ \hline 1125 \\ 225 \\ \hline 3375 \\ 15 \\ \hline 16875 \\ 3375 \\ \hline \end{array}$$

fourth power = 50625 Ans.

69. EXAMPLE.— 1.2^3 = what?

SOLUTION.— $1.2 \times 1.2 \times 1.2$,

or

$$\begin{array}{r} 1.2 \\ 1.2 \\ \hline 1.44 \\ 1.2 \\ \hline 288 \\ 144 \\ \hline \end{array}$$

cube = 1.728 Ans.

70. EXAMPLE.—What is the third power, or cube, of $\frac{3}{8}$?

SOLUTION.— $(\frac{3}{8})^3 = \frac{3}{8} \times \frac{3}{8} \times \frac{3}{8} = \frac{3 \times 3 \times 3}{8 \times 8 \times 8} = \frac{27}{512}$. Ans.

71. **Rule.—I.** *To raise a whole number or a decimal to any power, use it as a factor as many times as there are units in the exponent.*

II. *To raise a fraction to any power, raise both the numerator and denominator to the power indicated by the exponent.*

EXAMPLES FOR PRACTICE.

72. Raise the following to the powers indicated:

(*a*)	85^2.	(*a*)	7,225.
(*b*)	$(\frac{12}{13})^2$.	(*b*)	$\frac{144}{169}$.
(*c*)	6.5^2.	(*c*)	42.25.
(*d*)	14^4.	(*d*)	38,416.
(*e*)	$(\frac{3}{4})^3$.	(*e*)	$\frac{27}{64}$.
(*f*)	$(\frac{5}{6})^3$.	(*f*)	$\frac{125}{216}$.
(*g*)	$(\frac{7}{2})^3$.	(*g*)	$\frac{343}{8}$.
(*h*)	1.4^5.	(*h*)	5.37824.

Ans.

EVOLUTION.

73. **Evolution** is the reverse of involution. It is the process of finding the root of a number which is considered as a power.

74. The **square root** of a number is that number which, when used twice as a factor, produces the number.

Thus, 2 is the square root of 4, since 2×2, or $2^2 = 4$.

75. The **cube root** of a number is that number which, when used three times as a factor, produces the number.

Thus, 3 is the cube root of 27, since $3 \times 3 \times 3$, or $3^3 = 27$.

76. The **radical sign** $\surd$, when placed before a number, indicates that some root of that number is to be found.

77. The **index** of the root is a *small figure* placed *over* and to the *left* of the *radical sign*, to show what root is to be found.

Thus, $\sqrt[2]{100}$ denotes the *square root* of 100.
$\sqrt[3]{125}$ denotes the *cube root* of 125.
$\sqrt[4]{256}$ denotes the *fourth root* of 256, and so on.

78. When the square root is to be extracted, the index is generally omitted. Thus, $\surd 100$ indicates the square root of 100. Also, $\surd 225$ indicates the square root of 225.

SQUARE ROOT.

79. The *largest* number that can be written with *one* figure is 9, and $9^2 = 81$; the largest number that can be written with *two* figures is 99, and $99^2 = 9{,}801$; with *three* figures 999, and $999^2 = 998{,}001$; with *four* figures 9,999, and $9{,}999^2 = 99{,}980{,}001$, etc.

In *each* of the above it will be noticed that the square of the number contains just *twice* as many figures as the number.

In order to find the square root of a number, the first step is to find how many figures there will be in the root.

This is done by pointing off the number into *periods* of two figures each, *beginning at the right.* The number of periods will indicate the number of figures in the root.

Thus, the square root of 83,740,801 must contain 4 figures, since, pointing off the periods, we get 83′74′08′01, or 4 periods; consequently, there must be 4 figures in the root. In like manner, the square root of 50,625 must contain 3 figures, since there are (5′06′25) 3 periods.

80. EXAMPLE.—Find the square root of 31,505,769.

SOLUTION.—

				root	
			3 1′5 0′5 7′6 9	(5 6 1 3	Ans.
(*a*)	5				
	5	(*b*)	2 5		
(*d*)	1 0 0	(*c*)	6 5 0		
	6		6 3 6		
	1 0 6	(*e*)	1 4 5 7		
	6		1 1 2 1		
	1 1 2 0		3 3 6 6 9		
	1		3 3 6 6 9		
	1 1 2 1		0		
	1				
	1 1 2 2 0				
	3				
	1 1 2 2 3				

EXPLANATION.—Pointing off into periods of two figures each, it is seen that there are four figures in the root. Now, find the largest single number whose square is less than or equal to 31, the first period. This is evidently 5, since $6^2 = 36$, which is greater than 31. Write it to the right, as in long division, and also to the left as shown at (*a*). This is the first figure of the root. Now, multiply the 5 at (*a*) by the 5 in the root, and write the result under the first period, as shown at (*b*). Subtract, and obtain 6 as a remainder.

Bring down the next period, 50, and annex it to the remainder, 6, as shown at (*c*), which we call the **dividend.** Add the root already found to the 5 at (*a*), getting 10, and annex a cipher to this 10, thus making it 100, which we call the **trial divisor.** Divide the dividend (*c*) by the trial divisor (*d*) and obtain 6, which is probably the next figure of the root. Write 6 in the root, as shown, and also add it

to 100, the trial divisor, making it 106. This is called the **complete divisor.**

Multiply this by 6, the second figure in the root, and subtract the result from the dividend (*c*). The remainder is 14, to which annex the next period, making it 1,457, as shown at (*e*), which we call the **new dividend.** Add the second figure of the root to the complete divisor, 106, and annex a cipher, thus getting 1,120. Dividing 1,457 by 1,120, we get 1 as the next figure of the root. Adding this last figure of the root to 1,120, multiplying the result by it, and subtracting from 1,457, the remainder is 336.

Annexing the next and last period, 69, the result is 33,669. Now, adding the last figure of the root to 1,121, and annexing a cipher as before, the result is 11,220. Dividing 33,669 by 11,220, the result is 3, the fourth figure in the root. Adding it to 11,220 and multiplying the sum by it, the result is 33,669. Subtracting, there is no remainder; hence, $\sqrt{31,505,769} = 5,613$.

81. The square of any number wholly decimal always contains twice as many figures as the number squared. For example, $.1^2 = .01$, $.13^2 = .0169$, $.751^2 = .564001$, etc.

82. It will also be noticed that the number squared is always less than the decimal. Hence, if it be required to find the square root of a decimal, and the decimal has not an even number of figures in it, annex a cipher. The best way to determine the number of figures in the root of a decimal is to begin at the decimal point, and, going towards the *right*, point off the decimal into periods of two figures each. Then, if the last period contains but one figure, annex a cipher.

83. EXAMPLE.—What is the square root of .000576?

SOLUTION.—

		root
	2	.0 0′0 5′7 6 (.0 2 4 Ans.
	2	4
	40	176
	4	176
	44	0

EXPLANATION.—Beginning at the decimal point, and pointing off the number into periods of two figures each, it is seen that the first period is composed of ciphers; hence, the first figure of the root must be a cipher. The remaining portion of the solution should be perfectly clear from what has preceded.

84. If the number is not a perfect power, the root will consist of an interminable number of decimal places. The result may be carried to any required number of decimal places by annexing periods of two ciphers each to the number.

85. EXAMPLE.—What is the square root of 3? Find the result to five decimal places.

SOLUTION.—

			root
1		3.0 0'0 0'0 0'0 0'0 0 (	1.7 3 2 0 5 + Ans.
1		1	
20		200	
7		189	
27		1100	
7		1029	
340		7100	
3		6924	
343		176 0000	
3		173 2025	
3460		2797 5	
2		.	
3462			
2			
346400			
5			
346405			

EXPLANATION.—Annexing five periods of two ciphers each to the right of the decimal point, the first figure of the root is 1. To get the second figure we find that, in dividing 200 by 20, it is 10. This is evidently too large.

Trying 9, we add 9 to 20, and multiply 29 by 9; the result is 261, a result which is considerably larger than 200; hence, 9 is too large. In the same way it is found that 8 is also too large. Trying 7, 7 times 27 are 189, a result smaller than

200; therefore, 7 is the second figure of the root. The next two figures, 3 and 2, are easily found. The fifth figure in the root is a cipher, since the trial divisor, 34,640, is greater than the new dividend, 17,600. In a case of this kind we annex another cipher to 34,640, thereby making it 346,400, and bring down the next period, making the 17,600, 1,760,000. The next figure of the root is 5, and, as we now have five decimal places, we will stop.

The square root of 3 to five decimal places is, then, 1.73205+.

86. EXAMPLE.—What is the square root of .3 to five decimal places?

SOLUTION.—

		root
	5	.3 0'0 0'0 0'0 0'0 0'0 (.5 4 7 7 2+ Ans.
	5	25
	100	500
	4	416
	104	8400
	4	7609
	1080	79100
	7	76629
	1087	247100
	7	219084
	10940	28016
	7	
	10947	
	7	
	109540	
	2	
	109542	

EXPLANATION.—In the above example we annex a cipher to .3, making the first period .30, since every period of a decimal, as was mentioned before, must have two figures in it. The remainder of the work should be perfectly clear.

87. If it is required to find the square root of a mixed number, begin at the decimal point, and point off the periods both ways. The manner of finding the root will then be exactly the same as in the previous cases.

88. EXAMPLE.—What is the square root of 258.2449?

SOLUTION.—

		root
1	2'5 8.2 4'4 9 (1 6.0 7	Ans.
1	1	
20	158	
6	156	
26	22449	
6	22449	
3200	0	
7		
3207		

EXPLANATION.—In the above example, since 320 is greater than 224, we place a cipher for the third figure of the root, and annex a cipher to 320, making it 3,200. Then, bringing down the next period, 49, 7 is found to be the fourth figure of the root. Since there is no remainder, the square root of 258.2449 is 16.07.

89. Proof.—*To prove square root, square the result obtained. If the number is an exact power, the square of the root will equal it; if it is not an exact power, the square of the root will very nearly equal it.*

90. Rule.—I. *Begin at units place, and separate the number into periods of two figures each, proceeding from left to right with the decimal part, if there be any.*

II. *Find the greatest number whose square is contained in the first, or left-hand, period. Write this number as the first figure in the root; also, write it at the left of the given number.*

Multiply this number at the left by the first figure of the root, and subtract the result from the first period; then, annex the second period to the remainder.

III. *Add the first figure of the root to the number in the first column on the left, and annex a cipher to the result; this is the trial divisor. Divide the dividend by the trial divisor for the second figure in the root, and add this figure to the trial divisor to form the complete divisor. Multiply the complete divisor by the second figure in the root, and*

subtract this result from the dividend. (*If this result is larger than the dividend, a smaller number must be tried for the second figure of the root.*) *Now bring down the third period, and annex it to the last remainder for a new dividend. Add the second figure of the root to the complete divisor, and annex a cipher for a new trial divisor.*

IV. *Continue in this manner to the last period, after which, if any additional places in the root are required, bring down cipher periods, and continue the operation.*

V. *If at any time the trial divisor is not contained in the dividend, place a cipher in the root, annex a cipher to the trial divisor, and bring down another period.*

VI. *If the root contains an interminable decimal, and it is desired to terminate the operation at some point, say, the fourth decimal place, carry the operation one place further, and if the fifth figure is 5 or greater, increase the fourth figure by 1 and omit the sign +.*

91. Short Method.—If the number whose root is to be extracted is not an exact square, the root will be an interminable decimal. It is then usual to extract the root to a certain number of decimal places. In such cases, the work may be greatly shortened as follows: Determine to how many decimal places the work is to be carried, say 5, for example; add to this the number of places in the integral part of the root, say 2, for example, thus determining the number of figures in the root, in this case $5+2=7$. Divide this number by 2 and take the next higher number. In the above case, we have $7\div 2 = 3\frac{1}{2}$; hence, we take 4, the next higher number. Now extract the root in the usual manner until the same number of figures has been obtained as was expressed by the number obtained above, in this case 4. Then form the trial divisor in the usual manner, but omitting to add the cipher; divide the last remainder by the trial divisor as in long division, obtaining as many figures of the quotient as there are remaining figures of the root, in this case $7-4=3$. The remainder so obtained is the remaining figures of the root.

Consider the example in Art. **86.** Here there are 5 figures in the root. We therefore extract the root to 3 places in the usual manner, obtaining .547 for the first three root figures. The next trial divisor is 1,094 (with the cipher omitted) and the last remainder is 791. Then, $791 \div 1{,}094 = .723$, and the next two figures of the root are 72, the whole root being .54772 +. Always carry the division one place further than desired, and if the last figure is 5 or greater increase the preceding figure by 1. This method should not be used unless the root contains five or more figures.

If the last figure of the root found in the regular manner is a cipher, carry the process one place further before dividing as described above.

EXAMPLES FOR PRACTICE.

92. Find the square root of:

(*a*)	186,624.	(*a*)	432.
(*b*)	2,050,624.	(*b*)	1,432.
(*c*)	29,855,296.	(*c*)	5,464.
(*d*)	.0116964.	(*d*)	.1081 +.
(*e*)	198.1369.	(*e*)	14.0761.
(*f*)	994,009.	(*f*)	997.
(*g*)	2.375 to four decimal places.	(*g*)	1.5411.
(*h*)	1.625 to three decimal places.	(*h*)	1.275.
(*i*)	.3025.	(*i*)	.55.
(*j*)	.571428.	(*j*)	.7559 +.
(*k*)	.78125.	(*k*)	.8839.

Ans.

CUBE ROOT.

93. In the same manner as in the case of square root, it can be shown that the periods into which a number is divided, whose cube root is to be extracted, must contain three figures, except that the first, or left-hand, period of a whole or mixed number may contain one, two, or three figures.

94. EXAMPLE.—What is the cube root of 375,741,853,696?

SOLUTION.—

(1)	(2)	(3) root
7	49	375'741'853'696 (7216 Ans.
7	98	343
14	14700	32741
7	424	30248
210	15124	2493853
2	428	1557361
212	1555200	936492696
2	2161	936492696
214	1557361	0
2	2162	
2160	155952300	
1	129816	
2161	156082116	
1		
2162		
1		
21630		
6		
21636		

EXPLANATION.—Write the work in three columns, as follows: On the right, place the number whose cube root is to be extracted, and point it off into periods of three figures each. Call this column (3). Find the largest number whose cube is less than or equal to the first period, in this case 7. Write the 7 on the right as shown, for the first figure of the root, and also on the extreme left at the head of column (1). Multiply the 7 in column (1) by the first figure of the root, 7, and write the product, 49, at the head of column (2). Multiply the number in column (2) by the first figure of the root, 7, and write the product, 343, under the figures in the first period. Subtract and bring down the next period, obtaining 32,741 for the dividend. Add the first figure of the root to the number in column (1), obtaining 14, which call the *first correction.* Multiply the first correction by the first figure of the root, add the product to the number in column (2), and obtain 147. Add the first figure of the root to the first correction, and obtain 21, which call

the *second correction.* Annex *two* ciphers to the number in column (2), and obtain 14,700 for the trial divisor; also annex *one* cipher to the second correction, and obtain 210. Dividing the dividend by the trial divisor, we obtain $\frac{32,741}{14,700} = 2+$, and write the 2 as the second figure of the root. Add the 2 to the second correction, and obtain 212, which multiplied by the second figure of the root and added to the trial divisor, gives 15,124, the complete divisor. This last result multiplied by the second figure of the root and subtracted from the dividend, gives a remainder of 2,493. Annexing the third period, we obtain 2,493,853 for the new dividend. Adding the second figure of the root to the number in column (1) we get 214 as the new first correction; this multiplied by the second figure of the root and added to the complete divisor, gives 15,552. Adding the second figure of the root to the first new correction gives 216 as the second new correction. Annexing two ciphers to the number in column (2) gives 1,555,200, the new trial divisor. Annexing one cipher to the second new correction gives 2,160. Dividing the new dividend by the new trial divisor we obtain $\frac{2,493,853}{1,555,200} = 1+$, and write 1 as the third figure of the root. The remainder of the work should be perfectly clear from what has preceded.

95. In extracting the cube root of a decimal, proceed as above, taking care that each period contains *three* figures. Begin the pointing off at the decimal point, going towards the right. If the last period does not contain three figures, annex ciphers until it does.

96. EXAMPLE.—What is the cube root of .009129329 ?

SOLUTION.—

			root
2	4	.009'129'329	(.209
2	8	8	
4	120000	1129329	
2	5481	1129329	
600	125481	0	
9			
609			

EXPLANATION.—Beginning at the decimal, and pointing off as shown, the largest number whose cube is less than 9 is seen to be 2; hence, 2 is the first figure of the root. When finding the second figure, it is seen that the trial divisor, 1,200, is greater than the dividend; hence, write a cipher for the second figure of the root; bring down the next period to form the new dividend; annex two ciphers to the trial divisor to form a new trial divisor; also, annex one cipher to the 60 in the first column. Dividing the new dividend by the new trial divisor, we get $\frac{1129329}{120000} = 9+$, and write 9 as the third figure of the root. Complete the work as before.

97. EXAMPLE.—What is the cube root of 78,292.892952?

SOLUTION.—

			root
4	16	78′292.892′952	(42.78
4	32	64	
8	4800	14292	
4	244	10088	
120	5044	4204892	
2	248	3766483	
122	529200	438409952	
2	8869	438409952	
124	538069	0	
2	8918		
1260	54698700		
7	102544		
1267	54801244		
7			
1274			
7			
12810			
8			
12818			

EXPLANATION.—Since the above is a mixed number, begin at the decimal point and point off periods of three figures

each, in both directions. The first period contains but two figures, and the largest number whose cube is less than 78 is 4; consequently, 4 is the first figure of the root. The remainder of the work should be perfectly clear. When dividing the dividend by the trial divisor for the third figure of the root, the quotient was 8+, but, on trying it, it was found that 8 was too large, the complete divisor being considerably larger than the trial divisor. Therefore, 7 was used instead of 8.

98. EXAMPLE.—What is the cube root of 5 to five decimal places?

SOLUTION.—

			root
1	1	5.0 0 0′0 0 0′0 0 0′0 0 0′0 0 0 (	1.70997+
1	2	1	
2	300	4000	
1	259	3913	
30	559	87000000	
7	308	78443829	
37	8670000	8556171000	
7	45981	7889992299	
44	8715981	666178701000	
7	46062	614014317973	
5100	876204300	52164383027	
9	461511		
5109	876665811		
9	461592		
5118	877127403 00		
9	3590839		
51270	877163311 39		
9			
51279			
9			
51288			
9			
512970			
7			
512977			

EXPLANATION.—In the above example, we annex five periods of ciphers, of three ciphers each, to the 5 for the decimal part of the root, placing the decimal point between the 5 and the first cipher. Since it is easy to see that the next figure of the root will be 5, we increase the last figure by 1, obtaining 1.70998 for the correct root to 5 decimal places. Ans.

99. EXAMPLE.—What is the cube root of .5 to four decimal places?

SOLUTION.—

		root
7	49	.500′000′000′000(.7937+
7	98	343
14	14700	157000
7	1971	150039
210	16671	6961000
9	2052	5638257
219	1872300	1322743000
9	7119	1321748953
228	1879419	994047
9	7128	
2370	188654700	
3	166579	
2373	188821279	
3		
2376		
3		
23790		
7		
23797		

EXPLANATION.—In the above example, we annex two ciphers to the .5 to complete the first period, and three periods of three ciphers each. The cube root of 500 is 7; this we write as the first figure of the root. The remainder of the work should be perfectly plain from the explanations of the preceding examples.

100. EXAMPLE.—What is the cube root of .05 to four decimal places?

SOLUTION.—

			root
3	9	.05′000′000′000	(.3684+
3	18	27	
6	2700	23000	
3	576	19656	
90	3276	3344000	
6	612	3180032	
96	388800	163968000	
6	8704	162685504	
102	397504	1282496	
6	8768		
1080	406272,00		
8	44176		
1088	40671376		
8			
1096			
8			
11040			
4			
11044			

101. **Proof.**—*To prove cube root, cube the result obtained. If the given number is an exact power, the cube of the root will equal it; if not an exact power, the cube of the root will very nearly equal it.*

102. **Rule.**—I. *Arrange the work in three columns, placing the number whose cube root is to be extracted, in the third, or right-hand, column. Begin at units place, and separate the number into periods of three figures each, proceeding from the decimal point towards the right with the decimal part, if there is any.*

II. *Find the greatest number whose cube is not greater than the number in the first period. Write this number as the first figure of the root; also, write it at the head of the first column. Multiply the number in the first column by the first figure in the root, and write the result in the second column. Multiply the number in the second column by the first figure of the root; subtract the product from the first period, and*

annex the second period to the remainder for a new dividend; add the first figure of the root to the number in the first column for the first correction. Multiply the first correction by the first figure of the root, and add the product to the number in the second column. Add the first figure of the root to the first correction to form the second correction. Annex one cipher to the second correction and two ciphers to the last number in the second column; the last number in the second column is the trial divisor.

III. *Divide the dividend by the trial divisor to find the second figure of the root. Add the second figure of the root to the number in the first column, multiply the sum by the second figure of the root, and add the result to the trial divisor to form the complete divisor. Multiply the complete divisor by the second figure of the root, subtract the result from the dividend in the third column, and annex the third period to the remainder for a new dividend. Add the second figure of the root to the number in the first column to form the first correction; multiply the first correction by the second figure of the root, and add the product to the complete divisor. Add the second figure of the root to the first correction to form the second correction. Annex one cipher to the second correction and two ciphers to the last number in the second column to form the new trial divisor.*

IV. *If there are more periods to be brought down, proceed as before. If there is a remainder after the root of the last period has been found, annex cipher periods, and proceed as before. The figures of the root thus obtained will be decimals.*

V. *If the root contains an interminable decimal, and it is desired to terminate the operation at some point, say, the fourth decimal place, carry the operation one place further, and if the fifth figure is 5 or greater, increase the fourth figure by 1 and omit the sign +.*

103. Art. **91** can be applied to cube root (or any other root) as well as to square root. Thus, in the example, Art. **98**, there are to be $5+1=6$ figures in the root. Extracting the root in the usual manner to $6\div 2=3$,

say 4, figures, we get for the first four figures 1,709. The last remainder is 8,556,171, and the next trial divisor with the ciphers omitted is 8,762,043. Hence, the next two figures of the root are $8,556,171 \div 8,762,043 = .976$, say .98. Therefore, the root is 1.70998.

ROOTS OF FRACTIONS.

104. If the given number is in the form of a fraction, and it is required to find some root of it, the simplest and most exact method is to reduce the fraction to a decimal and extract the required root of the decimal. If, however, the numerator and denominator of the fraction are perfect powers, extract the required root of each separately, and write the root of the numerator for a new numerator, and the root of the denominator for a new denominator.

105. EXAMPLE.—What is the square root of $\frac{9}{64}$?

SOLUTION.— $\sqrt{\frac{9}{64}} = \frac{\sqrt{9}}{\sqrt{64}} = \frac{3}{8}$. Ans.

106. EXAMPLE.—What is the square root of $\frac{5}{8}$?

SOLUTION.—Since $\frac{5}{8} = .625$, $\sqrt{\frac{5}{8}} = \sqrt{.625} = .7906$. Ans.

107. EXAMPLE.—What is the cube root of $\frac{27}{64}$?

SOLUTION.— $\sqrt[3]{\frac{27}{64}} = \frac{\sqrt[3]{27}}{\sqrt[3]{64}} = \frac{3}{4}$. Ans.

108. EXAMPLE.—What is the cube root of $\frac{1}{4}$?

SOLUTION.—Since $\frac{1}{4} = .25$, $\sqrt[3]{\frac{1}{4}} = \sqrt[3]{.25} = .62996+$. Ans.

109. **Rule.**—*Extract the required root of the numerator and denominator separately; or, reduce the fraction to a decimal, and extract the root of the decimal.*

EXAMPLES FOR PRACTICE.

110. Find the cube root of:

(*a*) $\frac{27}{512}$.
(*b*) 2 to five decimal places.
(*c*) 4,180,769,192.462 to five decimal places.
(*d*) $\frac{87}{125}$.
(*e*) $\frac{3}{8}$.
(*f*) 513,229.783302144 to three decimal places.

Ans. (*a*) $\frac{3}{8}$. (*b*) 1.25992+. (*c*) 1,610.96238. (*d*) .8862+. (*e*) .7211+. (*f*) 80.064.

TO EXTRACT OTHER ROOTS THAN THE SQUARE AND CUBE ROOTS.

111. EXAMPLE.—What is the fourth root of 256?

SOLUTION.— $\sqrt{256} = 16.$

$\sqrt{16} = 4.$

Therefore, $\sqrt[4]{256} = 4.$ Ans.

In this example, $\sqrt[4]{256}$, the index is 4, which equals 2×2. The root indicated by 2 is the square root; therefore, the square root is extracted twice.

112. EXAMPLE.—What is the sixth root of 64?

SOLUTION.— $\sqrt{64} = 8.$

$\sqrt[3]{8} = 2.$

Therefore, $\sqrt[6]{64} = 2.$ Ans.

In this example, $\sqrt[6]{64}$, the index is 6, which equals 2×3. The root indicated by 3 is the cube root; therefore, the square and cube roots are extracted in succession.

113. **Rule.**—*Separate the index of the required root into its factors (2's and 3's), and extract, successively, the roots indicated by the several factors obtained. The final result will be the required root.*

114. EXAMPLE.—What is the sixth root of 92,873,580 to two decimal places?

SOLUTION.— $6 = 3 \times 2$. Hence, extract the cube root, and then extract the square root of the result. $\sqrt[3]{92,873,580} = 452.8601$, and $\sqrt{452.8601} = 21.28+$. Ans.

115. It matters not which root is extracted first, but it is probably easier and more exact to extract the cube root first.

EXAMPLES FOR PRACTICE.

116. Extract the

(*a*) Fourth root of 100.
(*b*) Fourth root of 3,049,800,625.
(*c*) Sixth root of 9,474,296,896.

Ans. (*a*) 3.16227+. (*b*) 235. (*c*) 46.

RATIO.

117. Suppose that it is desired to compare two numbers, say 20 and 4. If we wish to know how many times larger 20 is than 4, we divide 20 by 4 and obtain 5 for the quotient; thus, $20 \div 4 = 5$. Hence, we say that 20 is 5 times as large as 4, i. e., 20 contains 5 times as many units as 4. Again, suppose we desire to know what part of 20 is 4. We then divide 4 by 20 and obtain $\frac{1}{5}$; thus, $4 \div 20 = \frac{1}{5}$, or .2. Hence, 4 is $\frac{1}{5}$ or .2 of 20. This operation of comparing two numbers is termed *finding the ratio* of the two numbers. Ratio, then, is a comparison. It is evident that the two numbers to be compared must be expressed in the same unit; in other words, the two numbers must both be abstract numbers or concrete numbers of the same kind. For example, it would be absurd to compare 20 horses with 4 birds, or 20 horses with 4. Hence, **ratio** may be defined as a comparison between two numbers of the same kind.

118. A ratio may be expressed in three ways; thus, if it is desired to compare 20 and 4, and express this comparison as a ratio, it may be done as follows: $20 \div 4$, $20 : 4$, or $\frac{20}{4}$. All three are read *the ratio of 20 to 4*. The ratio of 4 to 20 would be expressed thus: $4 \div 20$, $4 : 20$, or $\frac{4}{20}$. The first method of expressing a ratio, although correct, is seldom or never used; the second form is the one oftenest met with, while the third is rapidly growing in favor, and is likely to supersede the second. The third form, called the fractional form, is preferred by modern mathematicians, and possesses great advantages to students of algebra and of higher mathematical subjects. The second form seems to be better adapted to arithmetical subjects, and is the one we shall ordinarily adopt. There is still another way of expressing a ratio, though seldom or never used in the case of a simple ratio like that given above. Instead of the colon, a straight vertical line is used; thus, $20 \mid 4$.

119. The **terms** of a ratio are the two numbers to be compared; thus, in the above ratio, 20 and 4 are the terms. When both terms are considered together they are called a **couplet**; when considered separately, the first term is called the **antecedent**, and the second term, the **consequent**. Thus, in the ratio 20 : 4, 20 and 4 form a couplet, and 20 is the antecedent, and 4, the consequent.

120. A ratio may be **direct** or **inverse**. The *direct ratio* of 20 to 4 is 20 : 4, while the *inverse ratio* of 20 to 4 is 4 : 20. The direct ratio of 4 to 20 is 4 : 20, and the inverse ratio is 20 : 4. An inverse ratio is sometimes called a **reciprocal** ratio. The **reciprocal** of a number is 1 divided by the number. Thus, the reciprocal of 17 is $\frac{1}{17}$; of $\frac{3}{8}$ is $1 \div \frac{3}{8} = \frac{8}{3}$; i. e., the reciprocal of a fraction is the fraction inverted. Hence, the inverse ratio of 20 to 4 may be expressed as 4 : 20 or as $\frac{1}{20} : \frac{1}{4}$. Both have equal values; for, $4 \div 20 = \frac{1}{5}$, and $\frac{1}{20} \div \frac{1}{4} = \frac{1}{20} \times \frac{4}{1} = \frac{1}{5}$.

121. The term **vary** implies a ratio. When we say that two numbers vary as some other two numbers, we mean that the ratio between the first two numbers is the same as the ratio between the other two numbers.

122. The **value** of a ratio is the result obtained by performing the division indicated. Thus, the value of the ratio 20 : 4 is 5; it is the quotient obtained by dividing the antecedent by the consequent.

123. By expressing the ratio in the fractional form, for example, the ratio of 20 to 4 as $\frac{20}{4}$, it is easy to see, from the laws of fractions, that if both terms be multiplied or both divided by the same number it will not alter the value of the ratio. Thus,

$$\frac{20}{4} = \frac{20 \times 5}{4 \times 5} = \frac{100}{20}; \text{ and } \frac{20}{4} = \frac{20 \div 4}{4 \div 4} = \frac{5}{1}.$$

124. It is also evident, from the laws of fractions, that multiplying the antecedent or dividing the consequent multiplies the ratio, and dividing the antecedent or multiplying the consequent divides the ratio.

125. When a ratio is expressed in words, as the ratio of 20 to 4, the first number named is always regarded as the antecedent and the second as the consequent, without regard to whether the ratio itself is direct or inverse. *When not otherwise specified, all ratios are understood to be direct.* To express an inverse ratio the simplest way of doing it is to express it as if it were a direct ratio, with the first number named as the antecedent, and then transpose the antecedent to the place occupied by the consequent and the consequent to the place occupied by the antecedent; or if expressed in the fractional form, invert the fraction. Thus, to express the inverse ratio of 20 to 4, first write it 20 : 4, and then, transposing the terms, as 4 : 20; or as $\frac{20}{4}$, and then inverting, as $\frac{4}{20}$. Or, the reciprocals of the numbers may be taken, as explained above. To **invert** a ratio is to transpose its terms.

EXAMPLES FOR PRACTICE.

126. What is the value of the ratio of:

(*a*)	98 : 49?	(*a*)	2.
(*b*)	\$45 : \$9?	(*b*)	5.
(*c*)	$6\frac{1}{4} : \frac{3}{4}$?	(*c*)	$12\frac{1}{2}$.
(*d*)	3.5 : 4.5?	(*d*)	$.77\frac{7}{9}$.
(*e*)	The inverse ratio of 76 to 19?	(*e*)	$\frac{1}{4}$.
(*f*)	The inverse ratio of 49 to 98?	(*f*)	2.
(*g*)	The inverse ratio of 18 to 24?	(*g*)	$1\frac{1}{3}$.
(*h*)	The inverse ratio of 9 to 15?	(*h*)	$1\frac{2}{3}$.
(*i*)	The ratio of 10 to 3, multiplied by 3?	(*i*)	10.
(*j*)	The ratio of 35 to 49, multiplied by 7?	(*j*)	5.
(*k*)	The ratio of 18 to 64, divided by 9?	(*k*)	$\frac{1}{32}$.
(*l*)	The ratio of 14 to 28, divided by 5?	(*l*)	$\frac{1}{10}$.

Ans.

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127. Instead of expressing the value of a ratio by a single number as above, it is customary to express it by

means of another ratio in which the consequent is 1. Thus, suppose that it is desired to find the ratio of the weights of two pieces of iron, one weighing 45 pounds and the other weighing 30 pounds. The ratio of the heavier to the lighter is then 45 : 30, an inconvenient expression. Using the fractional form, we have $\frac{45}{30}$. Dividing both terms by 30, the consequent, we obtain $\frac{1\frac{1}{2}}{1}$ or $1\frac{1}{2} : 1$. This is the same result as obtained above, for $1\frac{1}{2} \div 1 = 1\frac{1}{2}$, and $45 \div 30 = 1\frac{1}{2}$.

128. A ratio may be squared, cubed, or raised to any power, or any root of it may be taken. Thus, if the ratio of two numbers is 105 : 63, and it is desired to cube this ratio, the cube may be expressed as $105^3 : 63^3$. That this is correct is readily seen; for, expressing the ratio in the fractional form, it becomes $\frac{105}{63}$, and the cube is $\left(\frac{105}{63}\right)^3 = \frac{105^3}{63^3}$ $= 105^3 : 63^3$. Also, if it is desired to extract the cube root of the ratio $105^3 : 63^3$, it may be done by simply dividing the exponents by 3, obtaining 105 : 63. This may be proved in the same way as in the case of cubing the ratio. Thus, $105^3 : 63^3 = \left(\frac{105}{63}\right)^3$, and $\sqrt[3]{\left(\frac{105}{63}\right)^3} = \frac{105}{63} = 105 : 63$.

129. Since $\left(\frac{105}{63}\right)^3 = \left(\frac{5}{3}\right)^3$, it follows that $105^3 : 63^3$ $= 5^3 : 3^3$ (this expression is read, the ratio of 105 cubed to 63 cubed equals the ratio of 5 cubed to 3 cubed), and, hence, that the antecedent and consequent may both be multiplied or both divided by the same number, irrespective of any indicated powers or roots, without altering the value of the ratio. Thus, $24^2 : 18^2 = 4^2 : 3^2$. For, performing the operations indicated by the exponents, $24^2 = 576$ and $18^2 = 324$. Hence, $576 : 324 = 1\frac{7}{9}$ or $1\frac{7}{9} : 1$. Also, 4^2 $= 16$ and $3^2 = 9$; hence, $16 : 9 = 1\frac{7}{9}$ or $1\frac{7}{9} : 1$, the same result as before. Also, $24^2 : 18^2 = \frac{24^2}{18^2} = \left(\frac{24}{18}\right)^2 = \left(\frac{4}{3}\right)^2$ $= \frac{4^2}{3^2} = 4^2 : 3^2$.

The statement may be proved for roots in the same manner. Thus $\sqrt[3]{24^3} : \sqrt[3]{18^3} = \sqrt[3]{4^3} : \sqrt[3]{3^3}$. For, the $\sqrt[3]{24^3} = 24$ and $\sqrt[3]{18^3} = 18$; and, $24 : 18 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$. Also, $\sqrt[3]{4^3} = 4$ and $\sqrt[3]{3^3} = 3$; $4 : 3 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$.

If the numbers composing the antecedent and consequent have different exponents, or if different roots of those numbers are indicated, the operations above described cannot be performed. This is evident; for, consider the ratio of $4^2 : 8^3$. When expressed in the fractional form it becomes $\frac{4^2}{8^3}$, which cannot be expressed either as $\left(\frac{4}{8}\right)^2$ or as $\left(\frac{4}{8}\right)^3$, and, hence, cannot be reduced as described above.

PROPORTION.

130. **Proportion** is an equality of ratios, the equality being indicated by the double colon (::) or by the sign of equality (=). Thus, to write in the form of a proportion the two equal ratios, 8 : 4 and 6 : 3, which both have the same value, 2, we may employ one of the three following forms:

$$8 : 4 :: 6 : 3 \qquad (1)$$

$$8 : 4 = 6 : 3 \qquad (2)$$

$$\frac{8}{4} = \frac{6}{3} \qquad (3)$$

131. The first form is the one most extensively used, by reason of its having been exclusively employed in all the older works on mathematics. The second and third forms are being adopted by all modern writers on mathematical subjects, and, in time, will probably entirely supersede the first form. In this subject we shall adopt the second form, unless some statement can be made clearer by using the third form.

132. A proportion may be read in two ways. The old way to read the above proportion was—*8 is to 4 as 6 is to 3;* the new way is—*the ratio of 8 to 4 equals the ratio of 6 to 3.* The student may read it either way, but we recommend the latter.

133. Each ratio of a proportion is termed a **couplet.** In the above proportion, 8 : 4 is a couplet, and so is 6 : 3.

134. The numbers forming the proportion are called **terms**; and they are numbered consecutively from left to right, thus:

first second third fourth

$$8 : 4 = 6 : 3$$

Hence, in any proportion, the ratio of the first term to the second term equals the ratio of the third term to the fourth term.

135. The first and fourth terms of a proportion are called the **extremes,** and the second and third terms, the **means.** Thus, in the foregoing proportion, 8 and 3 are the extremes and 4 and 6 are the means.

136. A **direct proportion** is one in which both couplets are direct ratios.

137. An **inverse proportion** is one which requires one of the couplets to be expressed as an inverse ratio. Thus, 8 is to 4 inversely as 3 is to 6 must be written $8 : 4 = 6 : 3$; i. e., the second ratio (couplet) must be inverted.

138. Proportion forms one of the most useful sections of arithmetic. In our grandfathers' arithmetics, it was called "The rule of three."

139. Rule I.—*In any proportion, the product of the extremes equals the product of the means.*

Thus, in the proportion,

$$17 : 51 = 14 : 42.$$

$17 \times 42 = 51 \times 14$, since both products equal 714.

140. Rule II.—*The product of the extremes divided by either mean gives the other mean.*

EXAMPLE.—What is the third term of the proportion $17 : 51 = \quad : 42$?

SOLUTION.—Applying rule II, $17 \times 42 = 714$, and $714 \div 51 = 14$. Ans.

141. Rule III.—*The product of the means divided by either extreme gives the other extreme.*

EXAMPLE.—What is the first term of the proportion $\quad : 51 = 14 : 42$?

SOLUTION.—Applying rule III, $51 \times 14 = 714$, and $714 \div 42 = 17$. Ans.

142. When stating a proportion in which one of the terms is unknown, represent the missing term by a letter, as x. Thus, the last example would be written,

$$x : 51 = 14 : 42$$

and for the value of x we have $x = \frac{51 \times 14}{42} = 17$.

143. If the same operations (addition and subtraction excepted) be performed upon *all* the terms of a proportion, the proportion is not thereby destroyed. In other words, if all the terms of a proportion be (1) multiplied or (2) divided by the same number; (3) if all the terms be raised to the same power; (4) if the same root of all the terms be taken, or (5) if both couplets be inverted, the proportion still holds. We will prove these statements by a numerical example, and the student can satisfy himself by other similar ones. The fractional form will be used, as it is better suited to the purpose. Consider the proportion $8 : 4 = 6 : 3$. Expressing it in the third form, it becomes $\frac{8}{4} = \frac{6}{3}$. What we are to prove is that if any of the five operations enumerated above be performed upon all the terms of the proportion, the first fraction will still equal the second fraction.

1. Multiplying all the terms by any number, say 7, $\frac{8 \times 7}{4 \times 7} = \frac{6 \times 7}{3 \times 7}$; or $\frac{56}{28} = \frac{42}{21}$. Now $\frac{56}{28}$ evidently equals $\frac{42}{21}$, since the value of either ratio is 2, and the same is true of the original proportion.

2. Dividing all the terms by any number, say 7, $\frac{8 \div 7}{4 \div 7} = \frac{6 \div 7}{3 \div 7}$; or $\frac{\frac{8}{7}}{\frac{4}{7}} = \frac{\frac{6}{7}}{\frac{3}{7}}$. But $\frac{8}{7} \div \frac{4}{7} = 2$, and $\frac{6}{7} \div \frac{3}{7} = 2$ also, the same as in the original proportion.

3. Raising all the terms to the same power, say the cube, $\frac{8^3}{4^3} = \frac{6^3}{3^3}$. This is evidently true, since $\frac{8^3}{4^3} = \left(\frac{8}{4}\right)^3 = 2^3 = 8$, and $\frac{6^3}{3^3} = \left(\frac{6}{3}\right)^3 = 2^3 = 8$ also.

4. Extracting the same root of all the terms, say the cube root, $\frac{\sqrt[3]{8}}{\sqrt[3]{4}} = \frac{\sqrt[3]{6}}{\sqrt[3]{3}}$. It is evident that this is likewise true, since $\frac{\sqrt[3]{8}}{\sqrt[3]{4}} = \sqrt[3]{\frac{8}{4}} = \sqrt[3]{2}$, and $\frac{\sqrt[3]{6}}{\sqrt[3]{3}} = \sqrt[3]{\frac{6}{3}} = \sqrt[3]{2}$ also.

5. Inverting both couplets, $\frac{4}{8} = \frac{3}{6}$, which is true, since both equal $\frac{1}{2}$.

144. If both terms of either couplet be multiplied or both divided by the same number, the proportion is not destroyed. This should be evident from the preceding article, and also from Art. **123.** Hence, in any proportion, equal factors may be canceled from the terms of a couplet, before applying rule II or III. Thus, the proportion $45 : 9 = x : 7.1$, we may divide both terms of the first couplet by 9 (that is, cancel 9 from both terms), obtaining $5 : 1 = x : 7.1$, whence $x = 7.1 \times 5 \div 1 = 35.5$. (See Art. **129.**)

145. The principle of all calculations in proportion is this: *Three of the terms are always given, and the remaining one is to be found.*

146. EXAMPLE.—If 4 men can earn $25 in one week, how much can 12 men earn in the same time?

SOLUTION.—The required term must bear the same relation to the given term of the same kind, as one of the remaining terms bears to the other remaining term. We can then form a proportion by which the required term may be found.

The first question the student must ask himself in every calculation by proportion is:

"What is it I want to find?"

In this case it is dollars. We have two sets of men, one set earning $25, and we want to know how many dollars the other set earns. It is evident that the amount 12 men earn bears the same relation to the amount 4 men earn as 12 men bear to 4 men. Hence, we have the proportion, the amount 12 men earn is to $25 as 12 men are to 4 men, or, since either extreme equals the product of the means divided by the other extreme, we have

The amount 12 men earn : $25 :: 12 men : 4 men,

or the amount 12 men earn $= \frac{\$25 \times 12}{4} = \$75$. Ans.

Since it matters not which place x, or the required term, occupies, the problem could be stated in any of the following forms, the value of x being the same in each:

(a) \$25 : the amount 12 men earn = 4 men : 12 men; or the amount 12 men earn $= \frac{\$25 \times 12}{4}$, or \$75, since either mean equals the product of the extremes divided by the other mean.

(b) 4 men : 12 men = \$25 : the amount that 12 men earn; or the amount that 12 men earn $= \frac{\$25 \times 12}{4}$, or \$75, since either extreme equals the product of the means divided by the other extreme.

(c) 12 men : 4 men = the amount 12 men earn : \$25; or the amount that 12 men earn $= \frac{\$25 \times 12}{4}$, or \$75, since either mean equals the product of the extremes divided by the other mean.

147. If the proportion is an inverse one, first form it as though it were a direct proportion, and then invert one of the couplets.

EXAMPLES FOR PRACTICE.

148. Find the value of x in each of the following:

(a)	\$16 : \$64 :: x : \$4.	(a)	x = \$1.
(b)	x : 85 :: 10 : 17.	(b)	x = 50.
(c)	24 : x :: 15 : 40.	(c)	x = 64.
(d)	18 : 94 :: 2 : x.	(d)	$x = 10\frac{4}{9}$.
(e)	\$75 : \$100 = x : 100.	(e)	x = 75.
(f)	15 pwt. : x = 21 : 10.	(f)	$x = 7\frac{1}{7}$ pwt.
(g)	x : 75 yd. = \$15 : \$5.	(g)	x = 225 yd.

Ans.

1. If 75 pounds of lead cost \$2.10, what would 125 pounds cost at the same rate? Ans. \$3.50.

2. If A does a piece of work in 4 days and B does it in 7 days, how long will it take A to do what B does in 63 days? Ans. 36 days.

3. The circumferences of any two circles are to each other as their diameters. If the circumference of a circle 7 inches in diameter is 22 inches, what will be the circumference of a circle 31 inches in diameter? Ans. $97\frac{3}{7}$ inches.

INVERSE PROPORTION.

149. In Art. **137**, an inverse proportion was defined as one which required one of the couplets to be expressed as an inverse ratio. Sometimes the word *inverse* occurs in the statement of the example; in such cases, the proportion can

be written directly, merely inverting one of the couplets. But it frequently happens that only by carefully studying the conditions of the example, can it be ascertained whether the proportion is direct or inverse. When in doubt, the student can always satisfy himself as to whether the proportion is direct or inverse by first ascertaining what is required, and stating the proportion as a direct proportion. Then, in order that the proportion may be true, if the first term is smaller than the second term, the third term must be smaller than the fourth; or if the first term is larger than the second term, the third term must be larger than the fourth term. Keeping this in mind, the student can always tell whether the required term will be larger or smaller than the other term of the couplet to which the required term belongs. Having determined this, the student then refers to the example, and ascertains from its conditions whether the required term is to be larger or smaller than the other term of the same kind. If the two determinations agree, the proportion is direct, otherwise it is inverse, and one of the couplets must be inverted.

150. EXAMPLE.—If A's *rate* of doing work is to B's as 5 : 7, and A does a piece of work in 42 days, in what time will B do it?

SOLUTION.—The required term is the number of days it will take B to do the work. Hence, stating as a direct proportion,

$$5 : 7 = 42 : x.$$

Now, since 7 is greater than 5, x will be greater than 42. But, referring to the statement of the example, it is easy to see that B works faster than A; hence it will take B a less number of days to do the work than A. Therefore, the proportion is an inverse one, and should be stated

$$5 : 7 = x : 42$$

from which $x = \frac{5 \times 42}{7} = 30$ days. Ans.

Had the example been stated thus: The time that A requires to do a piece of work is to the time that B requires, as 5 : 7; A can do it in 42 days, in what time can B do it? it is evident that it would take B a longer time to do the work than it would A; hence, x would be greater than 42, and the proportion would be direct, the value of x being $\frac{7 \times 42}{5}$ = 58.8 days.

EXAMPLES FOR PRACTICE.

151. Solve the following:

1. If a pump which discharges 4 gal. of water per min. can fill a tank in 20 hr., how long will it take a pump discharging 12 gal. per min. to fill it? Ans. $6\frac{2}{3}$ hr.

2. If a pump discharges 90 gal. of water in 20 hr., in what time will it discharge 144 gal.? Ans. 32 hr.

3. The weight of any gas (the volume and pressure remaining the same) varies inversely as the absolute temperature. If a certain quantity of some gas weighs 2.927 lb. when the absolute temperature is 525°, what will the same volume of gas weigh when the absolute temperature is 600°, the pressure remaining the same? Ans. 2.561+ lb.

4. If 50 cu. ft. of air weigh 4.2 pounds when the absolute temperature is 562°, what will be the absolute temperature when the same volume weighs 5.8 pounds, the pressure being the same in both cases? Ans. 407°, very nearly.

POWERS AND ROOTS IN PROPORTION.

152. It was stated in Art. **128** that a ratio could be raised to any power or any root of it might be taken. A proportion is frequently stated in such a manner that one of the couplets must be raised to some power or some root of it must be taken. In all such cases, both terms of the couplet so affected *must be raised to the same power or the same root of both terms must be taken.*

153. EXAMPLE.—Knowing that the weight of a sphere varies as the cube of its diameter, what is the weight of a sphere 6 inches in diameter if a sphere 8 inches in diameter of the same material weighs 180 pounds?

SOLUTION.—This is evidently a direct proportion. Hence, we write

$$6^3 : 8^3 = x : 180.$$

Dividing both terms of the first couplet by 2^3 (see Art. **129**)

$$3^3 : 4^3 = x : 180, \text{ or } 27 : 64 = x : 180;$$

whence, $x = \frac{27 \times 180}{64} = 75\frac{15}{16}$ pounds. Ans.

EXAMPLE.—A sphere 8 inches in diameter weighs 180 pounds; what is the diameter of another sphere of the same material which weighs $75\frac{15}{16}$ pounds?

SOLUTION.—Since the weights of any two spheres are to each other as the cubes of their diameters, we have the proportion

$$180 : 75\tfrac{15}{16} = 8^3 : x^3.$$

The required term, x, must be cubed, because the other term of the couplet is cubed (see Art. **152**). But, $8^3 = 512$; hence,

$$180 : 75\tfrac{15}{16} = 512 : x^3, \text{ or } x^3 = \frac{75\frac{15}{16} \times 512}{180} = 216;$$

whence, $x = \sqrt[3]{216} = 6$ inches. Ans.

154. Since taking the same root of all the terms of a proportion does not change its value (Art. **143**), the above example might have been solved by extracting the cube root of all the numbers, thus obtaining $\sqrt[3]{180} : \sqrt[3]{75\frac{15}{16}} = 8 : x$; whence, $x = \dfrac{8 \times \sqrt[3]{75\frac{15}{16}}}{\sqrt[3]{180}} = 8 \times \sqrt[3]{\dfrac{75\frac{15}{16}}{180}} = 8\sqrt[3]{\dfrac{1,215}{2,880}} = 8\sqrt[3]{\dfrac{27}{64}}$ $= 8 \times \frac{3}{4} = 6$ inches. The process, however, is longer and is not so direct, and the first method is to be preferred.

155. If two cylinders have equal volumes, but different diameters, the diameters are to each other inversely as the square roots of their lengths. Hence, if it is desired to find the diameter of a cylinder that is to be 15 inches long, and which shall have the same volume as one that is 9 inches in diameter and 12 inches long, we write the proportion

$$9 : x = \sqrt{15} : \sqrt{12}.$$

Since neither 12 nor 15 are perfect squares, we square all the terms (Arts. **154** and **143**) and obtain

$$81 : x^2 = 15 : 12; \text{ whence, } x^2 = \frac{81 \times 12}{15} = 64.8,$$

and $x = \sqrt{64.8} = 8.05$ inches = diameter of 15-inch cylinder.

EXAMPLES FOR PRACTICE.

156. Solve the following examples:

1. The intensity of light varies inversely as the square of the distance from the source of light. If a gas jet illuminates an object 30 feet away with a certain distinctness, how much brighter will the object be at a distance of 20 feet? Ans. $2\frac{1}{4}$ times as bright.

2. In the last example, suppose that the object had been 40 feet from the gas jet; how bright would it have been, compared with its brightness at 30 feet from the gas jet? Ans. $\frac{9}{16}$ as bright.

3. When comparing one light with another, the intensities of their illuminating powers vary as the squares of their distances from the

source. If a man can just distinguish the time indicated by his watch, 50 feet from a certain light, at what distance could he distinguish the time from a light 3 times as powerful? Ans. 86.6+ feet.

4. The quantity of air flowing through a mine varies directly as the square root of the pressure. If 60,000 cubic feet of air flow per minute when the pressure is 2.8 pounds per square foot, how much will flow when the pressure is 3.6 pounds per square foot?

Ans. 68,034 cu. ft. per min., nearly.

5. In the last example, suppose that 70,000 cubic feet per minute had been required; what would be the pressure necessary for this quantity? Ans. 3.81+ lb. per sq. ft.

CAUSES AND EFFECTS.

157. Many examples in proportion may be more easily solved by using the principle of *cause and effect*. That which may be regarded as producing a change or alteration in something, or as accomplishing something, may be called a **cause**, and the change or alteration, or thing accomplished, as the **effect**.

158. *Like causes produce like effects.* Hence, when two causes of the same kind produce two effects of the same kind, the ratio of the causes equals the ratio of the effects; in other words the first cause is to the second cause as the first effect is to the second effect. Thus, in the question—if 3 men can lift 1,400 pounds, how many pounds can 7 men lift?—we call 3 men and 7 men the *causes* (since they accomplish something, viz., the lifting of the weight), the number of pounds lifted, viz., 1,400 pounds and x pounds, are the effects. If we call 3 men the first cause, 1,400 pounds is the first effect; 7 men is the second cause, and x pounds is the second effect. Hence, we may write

1st cause		*2d cause*		*1st effect*		*2d effect*
3	:	7	=	1,400	:	x

whence $x = \frac{7 \times 1{,}400}{3} = 3{,}266\frac{2}{3}$ pounds.

159. The principle of cause and effect is extremely useful in the solution of examples in compound proportion, as we shall now show.

COMPOUND PROPORTION.

160. All the cases of proportion so far considered have been cases of **simple proportion**; i. e., each term has been composed of but one number. There are many cases, however, in which two or all the terms have more than one number in them; all such cases belong to **compound proportion.** In all examples in compound proportion, both causes or both effects or all four consist of more than two numbers. We will illustrate this by an

EXAMPLE.—If 40 men earn $1,280 in 16 days, how much will 36 men earn in 31 days?

SOLUTION.—Since 40 men earn something, 40 men is a cause, and since they take 16 days in which to earn something, 16 days is also a cause. For the same reason 36 men and 31 days are also causes. The effects, that which is earned, are 1,280 dollars and x dollars. Then, 40 men and 16 days make up the first cause, and 36 men and 31 days make up the second cause. $1,280 is the first effect, and $\$x$ is the second effect. Hence, we write

$$\begin{matrix} \textit{1st cause} & & \textit{2d cause} & & \textit{1st effect} & & \textit{2d effect} \\ 40 & : & 36 & = & 1{,}280 & : & x \\ 16 & & 31 & & & & \end{matrix}$$

Now, instead of using the colon to express the ratio, we shall use the vertical line (see Art. **118**), and the above becomes

$$\begin{matrix} 40 \\ 16 \end{matrix} \Big| \begin{matrix} 36 \\ 31 \end{matrix} = 1{,}280 \Big| x.$$

In the last expression, the product of all the numbers included between the vertical lines must equal the product of all the numbers without them; i. e., $36 \times 31 \times 1{,}280 = 40 \times 16 \times x$.

$$\text{Or } x = \frac{36 \times 31 \times \overset{2}{\cancel{1280}}}{\underset{\cancel{16}}{\cancel{40}} \times \cancel{16}} = \$2{,}232. \quad \text{Ans.}$$

161. The above might have been solved by canceling factors of the numbers in the original proportion. For, if any number within the lines has a factor common to any number without the lines, that factor may be canceled from both numbers. Thus,

$$\begin{matrix} \cancel{40} \\ \cancel{16} \end{matrix} \Big| \begin{matrix} 36 \\ 31 \end{matrix} = \begin{matrix} 2 \\ \cancel{80} \\ \cancel{1280} \end{matrix} \Big| x.$$

16 is contained in 1,280, 80 times. Cancel 16 and 1,280, and write 80 above 1,280. 40 is contained in 80, 2 times. Cancel

40 and 80, and write 2 above 80. Now, since there are no more numbers that can be canceled, $x = 36 \times 31 \times 2 = \$2,232$, the same result as was obtained in the preceding article.

162. **Rule.**—*Write all the numbers forming the first cause in a vertical column, and draw a vertical line; on the other side of this line write in a vertical column all the numbers forming the second cause. Write the sign of equality to the right of the second column, and on the right of this form a third column of the numbers composing the first effect, drawing a vertical line to the right; on the other side of this line, write for a fourth column, the numbers composing the second effect. There must be as many numbers in the second cause as in the first cause, and in the second effect as in the first effect; hence, if any term is wanting, write x in its place. Multiply together all the numbers within the vertical lines, and also all those without the lines (canceling previously, if possible), and divide the product of those numbers which do not contain x by the product of the others in which x occurs, and the result will be the value of x.*

163. Example.—If 40 men can dig a ditch 720 feet long, 5 feet wide, and 4 feet deep in a certain time, how long a ditch 6 feet deep and 3 feet wide could 24 men dig in the same time?

Solution.—Here 40 men and 24 men are the causes, and the two ditches are the effects. Hence,

$$\begin{array}{c|c|c} & \cancel{3} & \\ & \cancel{18} & \\ & \cancel{720} & x \\ \cancel{40} & 24 = \quad 5 & \cancel{3} \\ & 4 & \cancel{6} \end{array} \text{ whence, } x = 24 \times 5 \times 4 = 480 \text{ feet. Ans.}$$

164. Example.—The volume of a cylinder varies directly as its length and directly as the square of its diameter. If the volume of a cylinder 10 inches in diameter and 20 inches long is 1,570.8 cubic inches, what is the volume of another cylinder 16 inches in diameter and 24 inches long?

Solution.—In this example, either the dimensions or the volumes may be considered the causes; say we take the dimensions for the causes. Then, squaring the diameters,

$$\begin{array}{c|c|c|c|c} 10^2 & 16^2 & & 100 & 256 \\ 20 & 24 = 1,570.8 & x, \text{ or } & \cancel{20} & \cancel{24} = 1,570.8 \quad x; \\ & & & 5 & 6 \end{array}$$

whence, $x = \dfrac{256 \times 6 \times 1,570.8}{5 \times 100} = 4,825.4976$ cubic inches. Ans.

165. EXAMPLE.—If a block of granite 8 ft. long, 5 ft. wide, and 3 ft. thick weighs 7,200 lb., what will be the weight of a block of granite 12 ft. long, 8 ft. wide, and 5 ft. thick?

SOLUTION.—Taking the weights as the effects, we have

$$\begin{array}{c|c|l} 8 & 4 & \\ 5 & 12 & \\ 3 & 8 = 7{,}200 & x, \text{ or } x = 4 \times 7{,}200 = 28{,}800 \text{ pounds. Ans.} \\ & 5 & \end{array}$$

166. EXAMPLE.—If 12 compositors in 30 days of 10 hours each set up 25 sheets of 16 pages each, 32 lines to the page, in how many days 8 hours long can 18 compositors set up, in the same type, 64 sheets of 12 pages each, 40 lines to the page?

SOLUTION.—Here compositors, days, and hours compose the causes, and sheets, pages, and lines the effects. Hence,

$$\begin{array}{c|c|c|l} 3 & 3 & 5 & 2 \\ 12 & 18 & 25 & 64 \\ & & 4 & \\ 30 & x = & 16 & 12, \text{ or } x = 3 \times 10 \times 2 = 60 \text{ days. Ans.} \\ 6 & & & 4 \\ 10 & 8 & 32 & 40 \\ & & & 5 \end{array}$$

167. In examples stated like that in Art. **164**, should an inverse proportion occur, write the various numbers as in the preceding examples, and then transpose from one side of the vertical line to the other side those numbers which are said to vary inversely.

EXAMPLE.—The centrifugal force of a revolving body varies directly as its weight, as the square of its velocity, and inversely as the radius of the circle described by the center of the body. If the centrifugal force of a body weighing 15 pounds is 187 pounds when the body revolves in a circle having a radius of 12 inches, with a velocity of 20 feet per second, what will be the centrifugal force of the same body when the radius is increased to 18 inches and the speed is increased to 24 feet per second?

SOLUTION.—Calling the centrifugal force the effect, we have

$$\begin{array}{c|c|l} 15 & 15 & \\ 20^2 & 24^2 = 187 & x. \\ 12 & 18 & \end{array}$$

Transposing 12 and 18 (since the radii are to vary inversely) and squaring 20 and 24,

$$\begin{array}{c|c|l} 15 & 15 & \\ & 2 & \\ 25 & 36 = 187 & x, \text{ or } x = \dfrac{12 \times 2 \times 187}{25} = 179.52 \text{ pounds. Ans.} \\ 400 & 576 & \\ 18 & 12 & \end{array}$$

EXAMPLES FOR PRACTICE.

168. Solve the following by compound proportion:

1. If 12 men dig a trench 40 rods long in 24 days of 10 hours each, how many rods can 16 men dig in 18 days of 9 hours each?

Ans. 36 rods.

2. If a piece of iron 7 feet long, 4 inches wide, and 6 inches thick weighs 600 pounds, how much will a piece of iron weigh that is 16 feet long, 8 inches wide, and 4 inches thick? Ans. $1{,}828\frac{4}{7}$ lb.

3. If 24 men can build a wall 72 rods long, 6 feet wide, and 5 feet high in 60 days of 10 hours each, how many days will it take 32 men to build a wall 96 rods long, 4 feet wide, and 8 feet high, working 8 hours a day? Ans. 80 days.

4. The horsepower of an engine varies as the mean effective pressure, as the piston speed, and as the square of the diameter of the cylinder. If an engine having a cylinder 14 inches in diameter develops 112 horsepower when the mean effective pressure is 48 pounds per square inch and the piston speed is 500 feet per minute, what horsepower will another engine develop if the cylinder is 16 inches in diameter, piston speed is 600 feet per minute, and mean effective pressure is 56 pounds per square inch? Ans. 204.8 horsepower.

5. Referring to the example in Art. **164**, what will be the volume of a cylinder 20 inches in diameter and 24 inches long?

Ans. 7,539.84 cubic inches.

6. Knowing that the product of $3 \times 5 \times 7 \times 9$ is 945, what is the product of $6 \times 15 \times 14 \times 36$? Ans. 45,360.

ELEMENTARY ALGEBRA
AND
TRIGONOMETRIC FUNCTIONS.

ELEMENTS OF ALGEBRA.

USE OF LETTERS.

1. In arithmetic, numbers are represented by the figures 1, 2, 3, 4, etc. There is no reason, however, why numbers may not be represented by other symbols, such as letters, if rules are provided for their use.

2. In algebra, numbers are represented by both figures and letters. It will be seen later that the use of letters often simplifies the solution of examples and shortens calculations.

3. The principal advantage of letters is that they are general in their meaning. Thus, unlike figures, the letter a does not stand for the number 1, the letter b for 2, c for 3, etc., but *any* letter may be taken to represent *any* number, it being only necessary that a letter shall always stand for the same number *in the same example.*

4. To illustrate this difference between letters and figures, we may take an example, as follows: If a farmer exchanges 20 bushels of oats, worth 40 cents per bushel,

for 8 bushels of wheat, what is the price of the wheat per bushel? A rule for solving this problem, and others like it, would be as follows: *Multiply the number of bushels of oats by the price per bushel, and divide the result by the number of bushels of wheat.*

This rule is *general*, because it tells us what to do with the number of bushels and with the prices of the oats and the wheat, *whatever they may be.*

A more concise way of stating this rule is to use letters in the same manner as in formulas. Thus:

Let a = number of bushels of oats;
b = price per bushel of oats;
c = number of bushels of wheat;
d = price per bushel of wheat.

Then, according to the rule,

$$\frac{\text{bushels of oats} \times \text{price per bushel}}{\text{bushels of wheat}} = \text{price of wheat},$$

or

$$\frac{a \times b}{c} = d.$$

In the example in question, $a = 20$, $b = 40$, and $c = 8$. Hence, writing for a, b, and c their values, 20, 40, and 8, d, the price per bushel of wheat $= \frac{20 \times 40}{8} = 100$. Here the expression $\frac{20 \times 40}{8}$ corresponds to $\frac{a \times b}{c}$, but this difference is to be noticed: $\frac{20 \times 40}{8}$ applies only to *this* example, and by performing the operations indicated only one answer can be obtained, while $\frac{a \times b}{c}$ is *general* in its application, in the same way that the rule previously given is general. That is, while a, b, and c stand for the numbers 20, 40, and 8 in *this* example, they may stand for other numbers in *another* example; hence, by writing their values in place of the letters, and by performing the operations indicated, the answer to any example of the same kind may be obtained. Consequently, while figures or combinations of figures always

represent the same numbers, letters are more general, and may represent any numbers, according to the conditions of the example.

5. An **equation** is a statement of equality between two expressions. Thus, $x+y = 8$ is an equation, and means that the sum of the numbers represented by x and y is equal to 8. Examples are solved in algebra by the aid of equations, in which numbers are represented both by letters and by figures. The following simple example will give an idea of the method of solution:

EXAMPLE.—If an iron rail 30 feet long is cut in two so that one part is four times as long as the other, how long is the shorter part?

SOLUTION.—Since any letter may represent any number,
Let x = the length of the shorter part.
Then, $4 \times x$ (written $4x$) = the length of the longer part.
But the sum of the two parts must equal the total length, 30 feet.
Hence, $x+4x = 30.$
Adding x and $4x$, $5x = 30.$
Whence, dividing by 5, $x = 6$ feet. Ans.

6. The student has probably noticed the similarity between an equation and a **formula.** All formulas are equations, and the same rules apply to both. An equation is not called a formula, however, unless it is a statement of a general rule.

7. **Algebra** treats of the equation and its use. Since the use of equations involves the use of letters, it will be necessary before considering equations to take up addition, subtraction, multiplication, etc. of expressions in which letters are used.

NOTATION.

8. The term **quantity** is used to designate any number that is to be subjected to mathematical processes. A quantity is strictly a concrete number; as, 6 books, 5 pounds, 10 yards. *Symbols* used to *represent* numbers, and expressions

containing two or more such symbols, as a, x, bd, 10, $(c+12)$, etc., are often called *quantities*, the term being a convenient one to use.

9. The **signs** $+$, $-$, $\times$, $\div$ are the same in algebra as in arithmetic. The sign of multiplication $\times$ is usually omitted, however, multiplication being indicated by simply writing the quantities together. Thus, abc means $a \times b \times c$; $2xy$ means $2 \times x \times y$. Evidently the sign cannot be omitted between two *figures*, as addition instead of multiplication would then be indicated. Thus, 24 means $20+4$ instead of 2×4.

10. A **coefficient** is a figure or letter prefixed to a quantity; it shows how many times the latter is to be taken. Thus, in the expression $4a$, 4 is the coefficient of a, and indicates that a is to be taken four times; that is, $4a$ is equal to $a+a+a+a$. When several quantities are multiplied together, any of them may be regarded as the coefficient of the others. Thus, in $6axy$, 6 is the coefficient of axy; $6a$, of xy; $6ax$, of y; etc. In general, however, when a coefficient is spoken of, the numerical coefficient only is meant, as the 6 above. When no numerical coefficient is written it is understood to be 1. Thus, cd is the same as $1cd$.

11. The **factors** of a quantity are the quantities which, when multiplied together, will produce it. Thus, 2, 3, and 3 are the factors of 18, since $2 \times 3 \times 3 = 18$; 2, a, and b are the factors of $2ab$, since $2 \times a \times b = 2ab$.

12. An **exponent** is a small figure placed at the right and a little above a quantity; it shows how many times the latter is to be taken *as a factor*. Thus, $4^3 = 4 \times 4 \times 4 = 64$, the exponent 3 showing that the number 4 is to be used three times as a factor; likewise $a^5 = aaaaa$. Any quantity written without an exponent is understood to have the exponent 1; thus, $b^1 = b$.

13. The difference between a coefficient and an exponent should be clearly understood. A coefficient *multiplies* the quantity which it precedes; it shows that the quantity is

to be *added to itself.* Thus, $3a = 3 \times a$, or $a + a + a$. An exponent indicates that a quantity is to be *multiplied by itself.* Thus, $a^3 = a \times a \times a$. A more complete definition of an exponent will be given later.

14. A **power** is the result obtained by taking a quantity two or more times *as a factor.* For example, 16 is the fourth power of 2, because 2 multiplied by itself until it has been taken four times as a factor produces 16; a^3 is the third power of a, because $a \times a \times a = a^3$.

15. A **root** of a quantity is one of its equal factors. Thus, 2 is the root of 4, 8, and 16, since $2 \times 2 = 4$, $2 \times 2 \times 2 = 8$, and $2 \times 2 \times 2 \times 2 = 16$, 2 being one of the equal factors in each case. In like manner, a is a root of a^2, a^3, a^5, etc. The symbol which denotes that the second, or square, root is to be extracted is $\sqrt{\ }$; it is called the **radical sign,** and the quantity under the sign is called the **radical.** For other roots the same symbol is used, but with a figure, called the *index* of the root, written above it to indicate the root. Thus, $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, etc. signify the square root, cube root, fourth root, etc. of a.

16. The use of the parenthesis, bracket, brace, and vinculum is explained in *Arithmetic.* These symbols are called **symbols of aggregation,** meaning that the quantities enclosed within them are aggregated, or collected, into one quantity.

17. The **terms** of an algebraic expression are those parts which are connected by the signs + and −. Thus, x^2, $-2xy$, and y^2 are terms of the expression $x^2 - 2xy + y^2$. When a term contains both figures and letters, the part consisting of letters is called the **literal** part of the term; thus, xy is the literal part of the term $2xy$.

18. **Like terms** are those which differ only in their numerical coefficients; all others are **unlike terms.** Thus, $2ab^2$ and $5ab^2$ are like terms; $5ab$ and $5ab^2$ are unlike terms, because one contains b and the other b^2.

19. A **monomial** is an expression consisting of only one term; as, $4abc$, $3x^2$, $2ax^3$, etc.

20. A **binomial** is an expression consisting of two terms; as, $a+b$, $2a+5b$, etc.

21. A **trinomial** is an expression consisting of three terms; as, $a^2+2ab+b^2$, $(a+x)^2-2(a+x)y+y^2$, etc , the expression $(a+x)$ being treated as one quantity.

22. A **polynomial** is an expression consisting of more than one term. The name is usually applied only to an expression consisting of four or more terms.

23. The polynomial $a+a^2b+2a^3-3a^4b-a^5$ is said to be arranged according to the *increasing powers* of a, because the exponents of a increase in each term from left to right, the exponent of the first a being 1 understood. (Art. **12.**) The polynomial $a^3b^3+ab^2+4a^4b+1$ is arranged according to the *decreasing powers* of b, the exponents of b decreasing in order from left to right.

24. The arrangement of the terms of a polynomial does not affect its value. Thus, $x^2+2xy+y^2$ has the same value as $2xy+y^2+x^2$, just as $2+6+4$ has the same value as $6+4+2$.

READING ALGEBRAIC EXPRESSIONS.

25. Quantities like a, x, b^2, etc. are read "a," "x," "b square," etc. In reading monomials in which multiplication is indicated, the word "times" is not used. Thus, abc is read "abc"; $7ad^2b^3$ is read "$7ad$ square b cube."

26. The polynomial $a+a^2b+2a^3-3a^4b-a^5$ is read "a, plus a square b, plus $2a$ cube, minus $3a$ fourth b, minus a fifth." Considerable care is required when reading expressions containing polynomials. Thus, if $4(a-b)$ were read "$4a$ minus b," the binomial $4a-b$ would be understood. It *should* be read "4 times $a-b$," or "4 times the parenthesis

a minus b," in which case it will be understood that 4 multiplies the whole quantity $a-b$, since the word "times" is not used with monomials. Again, $m(m^2+2mn+n^2)$ and $m(m^2+2mn)+n^2$ should each be so read that there can be no doubt as to whether the n^2 is to be multiplied by m or not.

Let the distinction to be made in reading the following be observed:

$$\sqrt{\frac{m+n}{x-y^2}} \text{ and } \sqrt{m+\frac{n}{x-y^2}}.$$

In the first case, the whole quantity $m+n$ is divided by $x-y^2$, and it would be clear to say, "the square root of the fraction $m+n$ over $x-y^2$." In the second case, where the n only is divided by $x-y^2$, it may be read, "the square root of the quantity, m plus the fraction n over $x-y^2$." The word "quantity" shows that the square root of the whole expression is taken, and the word "fraction" after "plus" shows that only the n is divided by $x-y^2$.

27. When a polynomial is affected by an exponent, it should be indicated clearly. Thus,

$$(3a-d^2)\ (3a-d)^2\ (3a-d^2)^2$$

should be read, "$3a-d$ square, times the square of $3a-d$, times the square of $3a-d$ square."

28. Sometimes expressions like A', B'', c', d'', C_1, a_2, etc. appear in formulas or elswhere in algebraic problems when it is desirable to have the same letter represent different quantities that are similar, or correspond to one another. The marks $'$, $''$, $'''$, $_1$, $_2$, etc. serve to distinguish the letters. The expressions are also used to designate similar or corresponding lines in geometrical figures. A', B'', C''', etc. are read "*a major prime, b major second, c major third,*" etc.; a', b'', c''', etc. are read "*a minor prime, b minor second, c minor third,*" etc.; a_1, B_2, C_3, d_4, etc. are read "*a minor sub-one, b major sub-two, c major sub-three, d minor sub-four,*" etc.

The words *major* and *minor* are used only when capitals and small letters are employed in the same problem. Otherwise they are dropped, and a', b_2, for example, are read "*a prime, b sub-two.*"

POSITIVE AND NEGATIVE QUANTITIES.

29. **Positive** and **negative** are terms applied to quantities of opposite character; as, money earned and money owed, water running into a tank and water running out, a distance up stream and a distance down stream, the height of a tower and the depth of a well, the pull on a lifting rope and the weight of the load, etc.

30. Positive quantities are preceded by the sign plus, as $+2xy$, $+ab$, etc., and negative quantities by the sign minus, as $-2xy$, $-ab$, etc. Thus, if money earned is $+\$50$, a like amount owed is $-\$50$. If the quantity of water running into a tank is denoted by $+a$, the same quantity running out should be denoted by $-a$.

31. It really does not matter which quantity be taken as positive and which as negative, so long as the characters of the positive and the negative quantity are opposite; but it is customary to call something gained positive and something lost negative. Thus, money earned is usually regarded as positive, money owed as negative; distance up, positive, distance down, negative.

32. The signs + and − may be used in two entirely different senses; heretofore we have used them exclusively as symbols of operation; thus, + placed between two quantities indicates that they are to be added, etc. In the distinction between positive and negative quantities, however, we denote the positive quantity by the sign + and the negative quantity by the sign −. Hence, under different circumstances, these signs may denote addition and subtraction, or they may denote positive and negative quantities. Suppose we write the expression $\$500 - \$200 = \$300$; the sign − in

this case indicates that the \$200 is subtracted from the \$500. Suppose, however, that a man has in his possession \$500 and owes \$200. The former amount we may denote by $+$ \$500, and the latter, since it is owed, by $-$\$200; in this case the sign $-$ before the \$200 indicates the negative character of the quantity. To find how much the man is worth we add the two, thus:

$$+\$500+(-\$200) = +\$300.$$

In this addition the second $+$ sign denotes the operation of addition, while the three signs immediately before the quantities denote the positive or negative character of the quantities.

33. It is usual to consider a quantity as increasing in a *positive* direction; *any positive quantity, no matter how small, is always considered greater than any negative quantity, no matter how large.*

The value of a negative quantity is conceived to *increase* as its numerical value *decreases*. A man who is \$10 in debt is better off than one who is \$50 in debt; and the man who has \$5 in the bank is better off than either. Thus, 5 is greater than -10, and -10 is greater than -50.

34. When writing algebraic expressions, if a positive term stands alone, or if the first term of an expression is positive, the plus sign is omitted, it being understood that the term is positive. Thus, $3a$ means the same as $+3a$, and $a-b$ the same as $+a-b$. The minus sign must never be omitted. Polynomials are usually written with a positive term first, and monomials with the letters arranged alphabetically.

EXAMPLES FOR PRACTICE.

35. Express the following algebraically:

1. Three x square y square, minus two cd into $a+b$.
Ans. $3x^2y^2-2cd(a+b)$.

2. The quantity m square plus two mn plus n square in parenthesis, times a square b cube c fourth. Ans. $(m^2+2mn+n^2)a^2b^3c^4$.

3. A, plus the square root of D, times the parenthesis X plus Y.

Ans. $A + \sqrt{D}(X + Y)$.

4. A, plus the radical D times the parenthesis X plus Y.

Ans. $A + \sqrt{D(X + Y)}$.

5. Ten x plus y, minus seven times the quantity x minus the fraction y over 4 in parenthesis, plus the fraction x square minus y square over two cd.

Ans. $10x + y - 7\left(x - \frac{y}{4}\right) + \frac{x^2 - y^2}{2cd}$.

When $a = 6$, $b = 5$, and $c = 4$, find the numerical values of:

6. $a^2 + 2ab + b^2$. Ans. $6^2 + 2 \times 6 \times 5 + 5^2 = 121$.

7. $2a^2 + 3bc - 5$. Ans. $72 + 60 - 5 = 127$.

8. $2ac^5 - a^2(a + b)$. Ans. 11,892.

9. $abc^2 + ab^2c - a^2bc$. Ans. 360.

When $x = 8$ and $y = 6$, what do the following equal:

10. $(x+y)(x-y) - \sqrt{\frac{x+y^2}{11}}$? Ans. $(8+6)(8-6) - \sqrt{\frac{8+6^2}{11}} = 26$.

11. $\sqrt{(x+y^2)(x^2+y)} - (x-y)(\sqrt[3]{x}+y)$? Ans. 39.5.

12. $\frac{x^2y^2}{x+y} + \frac{x^2y(x+y^2)}{\sqrt{3xy}}$? Ans. 1,572.57.

ADDITION AND SUBTRACTION.

PRELIMINARY IDEAS.

36. Suppose we take a point, as A on the line shown in Fig. 1, and lay off equal distances in opposite directions.

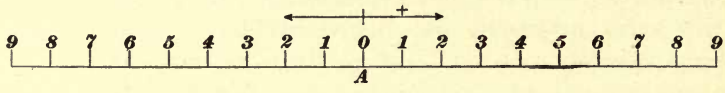

FIG. 1.

Now, let us call distances to the right +, or *positive*, and distances to the left —, or *negative*. Let us also call a movement to the right positive, and one to the left negative. Suppose the positive direction east, the negative west, and the distances to represent miles.

Suppose a man starts from A and walks east 6 miles, and after a pause walks 3 miles farther east. His distance from A is $+6+(+3) = +9$ miles, the plus sign being taken because the motion is in the positive direction. Suppose, however, the man starts from A and walks west 3 miles, and after a pause walks 5 miles farther west. His distance from A is 8 miles west of A, or 8 miles in a negative direction; that is, $-3+(-5) = -8$. As a third case, imagine the man to walk 6 miles east, and then turn around and walk 4 miles west. Counting 4 west from 6, we see that he would still be 2 miles east of A, or 2 miles in a positive direction; that is, $+6+(-4) = 2$. If, instead of walking back 4 miles, he had walked back 10 miles, we find by counting 10 miles west from 6 that he would have been 4 miles west of A, or 4 miles in a negative direction, $+6+(-10) = -4$. For reference, the above results are collected:

$$\begin{aligned} +6+(+3) &= +9 \\ -3+(-5) &= -8 \\ +6+(-4) &= +2 \\ +6+(-10) &= -4 \end{aligned}$$

The student should observe carefully that in each of these additions the signs immediately before the numbers denote their positive or negative character, while the + sign in front of the parenthesis denotes the operation of addition.

37. From these illustrations we have the following important principle: *If all the terms to be added are positive, the sum is positive; if all are negative, the sum is negative. If one term is positive and the other is negative, the sum has the sign of the numerically greater.* If there are several terms to be added, part of which are positive and part negative, the sum is positive if the sum of the positive terms is numerically greater than the sum of the negative terms.

When the terms have the same sign, the numerical sum is that which would be obtained if the signs were disregarded. Thus, in the first case above, the signs of 6 and 3 are the same, therefore the sum is numerically $6+3 = 9$; likewise

in the second, the signs of the 3 and the 5 are alike and the sum is numerically $3+5=8$. When, however, one term is positive and the other is negative, the sum is the numerical difference between the terms. Thus, in the third case above, the signs of 6 and 4 are different, and the sum, 2, is the numerical difference; likewise in the fourth case, the 6 and the 10 have different signs, and the sum, 4, is the numerical difference.

38. To add like terms containing letters, we simply add the coefficients, having regard for the proper signs, and annex the literal part. Thus, the sum of $6ax^2y$ and $3ax^2y$ is $9ax^2y$; the sum of $3ab$ and $-11ab$ is $-8ab$; and the sum of $9m^2n$, $3m^2n$, $-8m^2n$, and $2m^2n$ is $6m^2n$.

ADDITION OF MONOMIALS.

39. Like Quantities.—To add like quantities having the same sign:

Rule I.—*Add the coefficients, give the sum the common sign, and annex the common literal part.*

To add like quantities having different signs:

Rule II.—*Add the positive and the negative coefficients separately, and from the greater sum subtract the less. Give the remainder the sign of the greater sum, and annex the common literal part.*

EXAMPLE.—Find the sum of $-2abxy$, $-abxy$, $-3abxy$, and $-6abxy$.

SOLUTION.—The sum of the coefficients is 12 (remember that the coefficient of $-abxy$ is 1), and the common sign is $-$. The common literal part, $abxy$, annexed to these gives as the result $-12abxy$. (Rule I.)

EXAMPLE.—Combine xy^2, $-2xy^2$, $8xy^2$, and $-4xy^2$.

SOLUTION.—The sum of the coefficients of the positive terms is 9, and of the negative terms, 6. Their difference is 3, and the sign of the greater sum is $+$. The common literal part, xy^2, annexed to these gives as the result $3xy^2$. (Rule II.)

SUBTRACTION OF MONOMIALS.

40. Referring again to Fig. 1, suppose two men, C and D, start from point A and travel eastward. At the end of a certain time, C has walked 8 miles and D has walked 5 miles. Now, the distance between C and D is 3 miles; to pass from $+5$ to $+8$, we must walk $+3$ miles east, or in a *positive* direction; or $+8-(+5) = +3$. Observe that the minus sign here denotes subtraction, while the three plus signs denote that the three quantities which they precede are positive. The difference between 5 and 8 means how far, and in what direction we must go to pass from 5 to 8 or from 8 to 5. If we pass from $+5$ to $+8$, we move in a positive direction, and we say $+5$ from $+8$ is equal to $+3$. Suppose, however, we pass from $+8$ to $+5$; we move westward, or in a *negative* direction, a distance of 3 miles. Hence, we say that $+8$ from $+5$ is -3, or $+5-(+8) = -3$. We shall always consider the point we pass *from* as the subtrahend, or quantity to be subtracted, and the point we *approach* as the minuend, or quantity we subtract from.

Suppose C has walked 7 miles east and D has walked 4 miles west, how far apart are they? To pass from D to C we must travel 11 miles east, or in a positive direction. Therefore, $7-(-4) = +11$. To pass from C to D, we travel 11 miles west, or in a negative direction; $-4-(+7) = -11$.

The following exercises may be studied in connection with Fig. 1:

From $+3$ to $+8$ is $+5$, or $+8-(+3) = +5$.
From $+10$ to $+6$ is -4, or $+6-(+10) = -4$.
From -5 to $+4$ is $+9$, or $+4-(-5) = +9$.
From -9 to $-2 = +7$, or $-2-(-9) = +7$.
From -3 to $-7 = -4$, or $-7-(-3) = -4$.
From $+2$ to $-4 = -6$, or $-4-(+2) = -6$.

41. In every case, the difference is what must be added to the subtrahend to obtain the minuend. Thus, if I am 3 miles east of A, Fig. 1, how far must I go to be 8 miles east of A? Evidently 5 miles, since 5 miles added to 3 miles gives 8 miles. Similarly, the difference between -5 and

$+4$ is $+9$, since $+9$ must be added to -5 to obtain $+4$; that is, we must travel 9 miles east in passing from -5 to $+4$.

42. We have seen that if we add $+6$ and -4 we obtain $+2$ as the sum. (Art. **36.**) If we subtract $+4$ from $+6$, the difference is $+6-(+4) = +2$, since $+2$ must be added to $+4$ to make $+6$. If we add $+6$ and -10, the sum is -4. If we subtract $+10$ from $+6$, the difference is -4. Therefore, it appears that if we wish to subtract one quantity from another, we obtain the same result if we change the sign of the quantity to be subtracted and *add* it to the other quantity. Thus, $+7$ subtracted from $+12$ is the same as -7 added to $+12$, the result being 5 in either case. $-3mn$ subtracted from $4mn$ gives the same result as $+3mn$ added to $4mn$. The following exercises are given as illustrations:

$+5-(-2) = +5+(+2) = +7.$
$-6-(+3) = -6+(-3) = -9.$
$+2a-(+a) = +2a+(-a) = +a.$
$+5xy-(+15xy) = +5xy+(-15xy) = -10xy.$

43. To subtract like quantities:

Rule.—*Change the sign of the subtrahend, and proceed as in addition.*

EXAMPLE.—From $-3ab^2x$ take $7ab^2x$.

SOLUTION.—Changing the sign of the subtrahend, $7ab^2x$, and *adding*, we have $-3ab^2x+(-7ab^2x) = -10ab^2x$. Ans.

44. Unlike Quantities.—In arithmetic, unlike numbers, as 5 books and 3 dollars, cannot be added or subtracted. So, in algebra, unlike terms, as $3ab^2$, $4xy$, $2m$, etc., cannot be combined or subtracted, except by indicating the operations by signs.

Expressions in algebra are composed of quantities between which operations of addition, multiplication, etc. are indicated. The trinomial $m^2-2mn+n^2$, for example, is the indicated sum of m^2, $-2mn$, and n^2, *and it is to be considered*

as one quantity, in the same way that an arithmetical sum, obtained by actually performing the addition, is considered.

EXAMPLE.—What does $7cd^3 - 8cx - cd^3 + 6adx + 2cd^3$ equal?

SOLUTION.—In this case, part of the terms are like and part unlike. Combining like terms, $7cd^3 + 2cd^3 - cd^3 = 8cd^3$. Connecting the unlike terms with this result by their respective signs, we have as the final result $8cd^3 - 8cx + 6adx$. Ans.

EXAMPLE.—Subtract $2m - 3$ from $7m + 2xy$.

SOLUTION.—As in the example above, we have like and unlike terms. Subtracting like terms, Art. **43**, we have $7m - 2m = 5m$. We must now connect the unlike terms by their respective signs. Since -3 is in the subtrahend, its sign will be changed, giving us $5m + 2xy + 3$. Ans.

EXAMPLES FOR PRACTICE.

45. Find the sum of the following.

1. $-6a^2$, $2a^2$, $-5a^2$, $4a^2$, $-3a^2$, and a^2. Ans. $-7a^2$.
2. $2a^2b$, $-a^2b$, $11a^2b$, $-5a^2b$, $4a^2b$, and $-9a^2b$. Ans. $2a^2b$.
3. $2x^2$, $3xy$, $-x^2$, $8y^2$, $-5xy$, and $-7y^2$. Ans. $x^2 - 2xy + y^2$.

NOTE.—Combine like terms and connect with respective signs.

4. a^2bc, $-2ab^2c$, $3abc^2$, $-4a^2bc$, and $5ab^2c$.
Ans. $3ab^2c - 3a^2bc + 3abc^2$.

Solve the following:

5. From $17a$ take $-11a$. Ans. $28a$.
6. From $-11a$ take $17a$. Ans. $-28a$.
7. Subtract $5cd$ from $-4cd$. Ans. $-9cd$.
8. Subtract $-10b^2$ from $-10b^2$. Ans. 0.
9. What quantity added to $10xy$ will produce $-12xy$? Ans. $-22xy$.
10. What, then, does $10xy$ subtracted from $-12xy$ equal?
Ans. $-22xy$.

ADDITION AND SUBTRACTION OF POLYNOMIALS.

46. To add polynomials:

Rule.—*Write the expressions underneath one another, with like terms in the same vertical column. Add each column separately, and connect the sums by their proper signs.*

EXAMPLE.—Find the sum of $5a^2+6ac-3b^2-2xy$, $7ac-3a^2+4b^2+3xy$, and $4xy-5b^2+8ac-a^2$.

SOLUTION.—Writing like terms in the same vertical column, we have

$$\begin{array}{rl} & 5a^2+\ 6ac-3b^2-2xy \\ & -3a^2+\ 7ac+4b^2+3xy \\ & -\ a^2+\ 8ac-5b^2+4xy \\ \hline \textit{sum} & a^2+21ac-4b^2+5xy. \quad \text{Ans.} \end{array}$$

EXAMPLE.—Find the sum of $a^2x-ax^2-x^2$, $ax-x^2-a^2$, $-2a^2-2a^2x-2ax^2$, and $3a^2-3a^2x+3ax^2$.

SOLUTION.—

$$\begin{array}{rl} & a^2x-\ ax^2-\ x^2 \\ & \qquad\quad -\ x^2-\ a^2+ax \\ & -2a^2x-2ax^2 \qquad -2a^2 \\ & -3a^2x+3ax^2 \qquad +3a^2 \\ \hline \textit{sum} & -4a^2x+0 \qquad -2x^2+0 \quad +ax \\ & = ax-4a^2x-2x^2. \quad \text{Ans. (Arts. 23 and 24.)} \end{array}$$

47. To subtract one polynomial from another:

Rule.—*Write the subtrahend underneath the minuend, with like terms in the same vertical column. Change the sign of each term of the subtrahend, and proceed as in addition.*

EXAMPLE.—From $3ac-2b$ subtract $ac-b-d$.

SOLUTION.—

$$\begin{array}{rl} & 3ac-2b \\ & -\ ac+\ b+d, \text{ subtrahend with signs changed.} \\ \hline \textit{difference} & 2ac-\ b+d. \quad \text{Ans.} \end{array}$$

EXAMPLE.—From $2x^3-3x^2y+2xy^2$ subtract $x^3-xy^2+y^3$.

SOLUTION.—

$$\begin{array}{rl} & 2x^3-3x^2y+2xy^2 \\ & -\ x^3\qquad +\ xy^2-y^3, \text{ subtrahend with signs changed.} \\ \hline \textit{difference} & x^3-3x^2y+3xy^2-y^3. \quad \text{Ans.} \end{array}$$

EXAMPLES FOR PRACTICE.

48. Find the sum of the following:

1. $ax+2bx+4by-3ay$, $2ax+bx+2ay-by$, and $4ax+3by$.
   Ans. $7ax+3bx+6by-ay$.
2. $a-x+4y-3z+w$, $z+3a-2x-y-w$, and $x+y+z$.
   Ans. $4a-2x+4y-z$.

3. $2a-3b+4d$, $2b-3d+4c$, $2d-3c+4a$, and $2c-3a+4b$.
Ans. $3a+3b+3c+3d$.

4. $6x-3y+7m$, $2n-x+y$, $2y-4x-5m$, and $m+n-y$.
Ans. $x-y+3m+3n$.

Solve the following:

5. From $7a+5b-3c$ take $a-7b+5c-4$.
Ans. $6a+12b-8c+4$.

6. From $3m-5n+r-2s$ take $2r+3n-m-5s$.
Ans. $4m-8n-r+3s$.

7. Subtract $2x-2y+2$ from $y-x$. Ans. $3y-3x-2$.

8. Subtract $3x^3+4x^2y-7xy^2+y^3-xy^3$ from $5x^3+x^2y-6xy^2+y^3$.
Ans. $2x^3-3x^2y+xy^2+xy^3$.

SYMBOLS OF AGGREGATION.

49. Parentheses, brackets, etc. being used to enclose expressions that are to be treated as one quantity, the sign before the symbol affects the *entire expression*, not the first term only. Thus, $-(a^2-2ab+b^2)$ signifies that all the terms are to be subtracted from what precedes, not a^2 only.

50. When combining the terms of any expression without parenthesis, we proceed as in addition of monomials. When we have a parenthesis preceded by a minus sign, we must consider the expression within the parenthesis as a subtrahend, and change all signs before removing the parenthesis.

If, on the contrary, the sign of the parenthesis is plus, we may remove the symbols, but we must not change the signs of the expression, because we do not change the signs of an expression to be added.

51. When a quantity is enclosed by a parenthesis, the first term is understood to have the plus sign, unless the minus sign is given; thus, in the expression $-(8x+5-2b)$, the minus sign refers to the whole quantity. The sign of $8x$ is $+$, and the expression if written in full would be $-(+8x+5-2b)$.

EXAMPLE.—Remove the parenthesis from $4c-(3a+4ab-d)$.

SOLUTION.—Changing the sign of each enclosed term, and remembering that the sign of $3a$ is $+$, understood, we have as the result $4c-3a-4ab+d$. Ans.

EXAMPLE.—Remove the parentheses from $4a-5x-(a-4x)+(x-8a)$.

SOLUTION.— $4a-5x-(a-4x)+(x-8a)=4a-5x-a+4x+x-8a$. Adding the like terms, we have

$$\begin{array}{r} 4a-5x \\ -\ a+4x \\ -8a+\ x \\ \hline -5a+0 \end{array} = -5a. \quad \text{Ans.}$$

52. Symbols of aggregation will often be found enclosing others. In such cases they may be removed in succession, *always beginning with the innermost pair.*

EXAMPLE.—Remove all the symbols of aggregation from

$$6a-\{b-[7cd-4a+(2cd-\overline{a-b})]\}.$$

SOLUTION.—We first remove the vinculum. This being in effect the same as the parenthesis, the minus sign before the a indicates that $+a$ and $-b$ are to be subtracted.

Hence, we have

$$6a-\{b-[7cd-4a+(2cd-a+b)]\}.$$

Removing the parenthesis we have

$$6a-\{b-[7cd-4a+2cd-a+b]\}.$$

This, with the brackets removed, is equal to

$$6a-\{b-7cd+4a-2cd+a-b\},$$

which, in turn, is equal to

$$6a-b+7cd-4a+2cd-a+b.$$

Combining like terms,

$$6a-4a-a-b+b+7cd+2cd=a+9cd. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

53. Remove the parentheses from the following:

1. $-(2mn-m^2-n^2)$. Ans. $m^2-2mn+n^2$.
2. $1-(-b+c+3)$. Ans. $b-c-2$.
3. $5a-4b+3c-(-3a+2b-c)$. Ans. $8a-6b+4c$.
4. $3x-(2x-5)+(7-x)$. Ans. 12.

Remove the symbols of aggregation from the following:

5. $m-[4n-k-(m+n-2k)]$. Ans. $2m-3n-k$.

6. $5x-(2x-3y)-(x+5y)$. Ans. $2x-2y$.
7. $3a-[7a-(5a-b-a)]-(-a-4b)$. Ans. $a+3b$.
8. $3x+\{2y-[5x-(3y+\overline{x-4y})]\}$. Ans. $y-x$.
9. $100x-\{200x-[500x-(-100x)-300x]-400x\}$. Ans. $600x$.
10. $7cx-\{4cy-[(4cx+3cy)+cy-cx]\}$. Ans. $10cx$.

NOTE.—Observe that the sign before the parenthesis is + understood.

MULTIPLICATION.

PRELIMINARY IDEAS.

54. In algebra, **multiplication** is often indicated only, and the final answer is frequently nothing more than the symbols so written that it is shown they should be multiplied when their numerical values are substituted. Thus, the product of m and n is mn, the absence of a sign between the two quantities denoting the operation of multiplication. If we multiply $2a$ by $3b$, the four factors which form the product are 2, a, 3, and b; hence, $2a\times 3b = 2\times a\times 3\times b$. Since we are at liberty to arrange the factors in any order, we have

$$2a\times 3b = 2\times 3\times a\times b = 6\times a\times b = 6ab.$$

Hence, in finding the product of two quantities, *the coefficients are multiplied together and prefixed to the literal factors.*

55. When two quantities are multiplied together, *the sign of the product is positive if the two quantities have the same sign; and it is negative if the quantities have opposite signs.*

Thus,
$$(+5n)\times(+3m) = +15mn.$$
$$(-5n)\times(+3m) = -15mn.$$
$$(-5n)\times(-3m) = +15mn.$$
$$(+5n)\times(-3m) = -15mn.$$

56. Assume that the $3m$ above is the multiplier, and that $5n$ is the multiplicand. The multiplier must be considered as an abstract number, and simply shows how many

times the multiplicand is to be taken; thus, $3m$ may represent 15, 21, 30, etc., depending upon the value of m. The multiplicand, $5n$, may represent any concrete number; as, \$20, 60 feet, 100 miles. The sign of the multiplicand shows the character of the quantity; thus, if $+5n$ represents \$20, $-5n$ may be taken to represent a debt of \$20. The sign of the multiplier, on the other hand, is a symbol of operation, and shows how the result is to be treated after it is obtained. If the multiplier is positive, the result obtained is to be added when taken in connection with other quantities; the multiplier is given the negative sign to show that the result is to be subtracted.

Multiplying $+5n$ by $3m$, considering the multiplier simply as an abstract number, and disregarding its sign, we obtain $+15mn$. Similarly, $-5n$ multiplied by $3m$ gives $-15mn$. The product in each case has the same character as the multiplicand. Now, if the multiplier has the positive sign, the product is to be added when taken in connection with other quantities, and, therefore, the signs remain as above. That is, $(+5n)\times(+3m) = +15mn$, and $(-5n)\times(+3m) = -15mn$.

If, on the other hand, the multiplier has the negative sign, the product is to be subtracted, and, therefore, its sign must be changed when combined with other quantities. Thus, when the multiplier, $3m$, has the negative sign, we have $(+5n)\times(-3m) = -15mn$ and $(-5n)\times(-3m) = +15mn$.

57. Exponents.—It has been shown that $a^2 = a\times a$ and $a^3 = a\times a\times a$. The product $a^2\times a^3$, or $a^2a^3 = a\times a\times a\times a\times a = a^5$. The exponent 2 shows that a is used twice as a factor, and the exponent 3 shows that a is used three times as a factor. In the product, a must be used five times as a factor, or the exponent of a is 5. *The exponent of the product is the sum of the exponents of the factors.*

Thus,

$$a^4\times a^3 = a^{4+3} = a^7.$$
$$c^2\times c^4 = c^{2+4} = c^6.$$
$$m\times m^2\times m^3 = m^{1+2+3} = m^6.$$
$$n^2\times n^4\times n^5\times n^3 = n^{14}.$$

MULTIPLICATION OF MONOMIALS.

58. Rule.—*To the product of the coefficients annex the letters of both factors; give each letter an exponent equal to the sum of the exponents of that letter.*

Make the sign of the product plus, when the signs of the factors are alike; and minus, when they are unlike.

EXAMPLE.—Multiply $4a^2b$ by $-5a^3bc$.

SOLUTION.—The product of the coefficients is 20, and the letters to be annexed are a, b, and c. The new exponent of a is 5, and of b, 2, since $a^{2+3} = a^5$, and $b^{1+1} = b^2$. The sign of the product is minus, since the two factors have different signs. Hence, $4a^2b \times -5a^3bc = -20a^5b^2c$. Ans.

59. When there are more than two factors, we have simply three or more examples in multiplication to solve in succession, each to be performed by the foregoing rule.

EXAMPLE.—Find the continued product of $6x^2yz^3$, $-9x^2y^2z^2$, and $-3x^4yz$.

SOLUTION.— $6x^2yz^3 \times -9x^2y^2z^2 = -54x^{2+2}y^{1+2}z^{3+2}$, or $-54x^4y^3z^5$. Now, multiplying this product by $-3x^4yz$, we have $-54x^4y^3z^5 \times -3x^4yz = 162x^8y^4z^6$. Ans.

EXAMPLES FOR PRACTICE.

60. Find the product of:

1. a^3b^2 and $-5abd$. Ans. $-5a^4b^3d$.
2. $-7xy$ and $-7x^2y^2$. Ans. $49x^3y^3$.
3. $-15m^5n^6$ and $3mn$. Ans. $-45m^6n^7$.
4. $3a(x-y)^2$ and $2a^2(x-y)$. Ans. $6a^3(x-y)^3$.

SUGGESTION.—Treat the $(x-y)$ as though it were a single letter.

5. Find the continued product of $2a^3m^2x$, $-3a^2mx^3$, and $4am^3x^2$. Ans. $-24a^6m^6x^6$.
6. What does $-a^2bn \times -2cdn \times -3bdc^2 \times -2acn^2$ equal? Ans. $12a^3b^2c^4d^2n^4$.

MULTIPLICATION OF POLYNOMIALS.

61. When one of the factors is a monomial:

Rule.—*Multiply each term of the polynomial by the monomial, and connect the separate products by their proper signs.*

EXAMPLE.—Find the product of $-9a^5+3a^3b^2-4a^2b^3-b^5$ and $-3ab^4$.

SOLUTION.—

$$\begin{array}{l} -9a^5+3a^3b^2-4a^2b^3-b^5 \\ \quad -3ab^4 \\ \hline 27a^6b^4-9a^4b^6+12a^3b^7+3ab^9. \quad \text{Ans.} \end{array}$$

62. When both factors are polynomials:

Rule.—*Multiply each term of one polynomial by each term of the other, and add the partial products.*

EXAMPLE.—Multiply $6a-4b$ by $4a-2b$.

SOLUTION.—Write the multiplier under the multiplicand, and begin to multiply *at the left* instead of at the right, as in arithmetic, since polynomials are always written and read from the left, and there are no numbers to carry.

$$\begin{array}{lll} & 6a-4b & (1) \\ & 4a-2b & \\ \hline \text{Multiplying (1) by } 4a \text{ gives} & 24a^2-16ab & (2) \\ \text{Multiplying (1) by } -2b \text{ gives} & \quad -12ab+8b^2 & (3) \\ \hline \text{Adding (2) and (3) gives} & 24a^2-28ab+8b^2. & \text{Ans.} \end{array}$$

It will be noticed that the like terms, $-16ab$ and $-12ab$, are written under each other, so that it will be easier to add them.

EXAMPLE.—Multiply $x^3-x+1+x^2$ by $1-x^2+x$.

SOLUTION.—With a view to bringing like terms in the same columns, arrange both multiplicand and multiplier either according to the increasing or the decreasing powers of the same letter. (Art. **23.**) Arranging in this case according to the increasing powers of x, we have

$$\begin{array}{lll} & 1-x+x^2+x^3 & (1) \\ & 1+x-x^2 & \\ \hline \text{Multiplying (1) by 1 gives} & 1-x+x^2+x^3 & (2) \\ \text{Multiplying (1) by } +x \text{ gives} & \quad x-x^2+x^3+x^4 & (3) \\ \text{Multiplying (1) by } -x^2 \text{ gives} & \quad -x^2+x^3-x^4-x^5 & (4) \\ \hline \text{Adding (2), (3), and (4) gives} & 1 \quad -x^2+3x^3 \quad -x^5. & \text{Ans.} \end{array}$$

63. Multiplication is frequently indicated by enclosing each of the quantities to be multiplied in a parenthesis. The sign of multiplication is not placed between the parentheses, multiplication being understood. When the quantities are multiplied together, the expression is said to be *expanded.*

For example, in the expression $(m-2n)(2m-n)$, the

binomial $m-2n$ is to be multiplied by the binomial $2m-n$. Performing the multiplication, the product is $2m^2-5mn+2n^2$, which is the expanded form of the expression.

EXAMPLES FOR PRACTICE.

64. Multiply the following:

1. $x^2+2xy+y^2$ by $x+y$. Ans. $x^3+3x^2y+3xy^2+y^3$.
2. $3ab^2m^3+4a^2b-2$ by $a^6b^7m^8$. Ans. $3a^7b^9m^{11}+4a^8b^8m^8-2a^6b^7m^8$.
3. c^2-d^2 by c^2+d^2. Ans. c^4-d^4.
4. $x^4+x^2y^2+y^4$ by x^2-y^2. Ans. x^6-y^6.
5. $3a^2-7a+4$ by $2a^2+9a-5$. Ans. $6a^4+13a^3-70a^2+71a-20$.

Expand the following:

6. $(2a-3c)(4-3a)$. Ans. $8a-12c-6a^2+9ac$.
7. $(x+2)(x-2)(x^2+4)$. Ans. x^4-16.
8. $[x(x^2-y^2)-2][x(x^2+y^2)+2]$.

NOTE.—The expressions in the brackets reduce to x^3-xy^2-2 and x^3+xy^2+2. The product of these is $x^6-x^2y^4-4xy^2-4$. Ans.

THREE IMPORTANT EXAMPLES.

65. Let a and b be *any* two quantities; we wish to find the forms of the following products:

$$(a+b)^2,\ (a-b)^2,\ \text{and}\ (a+b)(a-b).$$

By actual multiplication we find

$$(a+b)^2 = a^2+2ab+b^2. \qquad (1.)$$

$$(a-b)^2 = a^2-2ab+b^2. \qquad (2.)$$

$$(a+b)(a-b) = a^2-b^2. \qquad (3.)$$

Hence:

66. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and the second, plus the square of the second.*

The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and the second, plus the square of the second.

The product of the sum and difference of two quantities is equal to the difference of their squares.

67. The foregoing statements should be committed to memory, since their use will frequently save tedious calculation. *The student is advised to practice until he is certain that he knows them perfectly.*

EXAMPLE.—Square $3x^2 + 5$.

SOLUTION.—The square of the first term is $3x^2 \times 3x^2 = 9x^4$; twice the product of the terms is $30x^2$; and the square of the last term is 25. Hence, by formula **1**, letting $a = 3x^2$ and $b = 5$,

$$(3x^2 + 5)^2 = 9x^4 + 30x^2 + 25. \quad \text{Ans.}$$

EXAMPLE.—Square $4cd - x$.

SOLUTION.—The square of the first term is $16c^2d^2$; twice the product of the first and the second is $-8cdx$; and the square of the last term is x^2. Hence, by formula **2**, letting $a = 4cd$ and $b = x$,

$$(4cd - x)^2 = 16c^2d^2 - 8cdx + x^2. \quad \text{Ans.}$$

EXAMPLE.—Expand $(x^2 + 3)(x^2 - 3)$. (See Art. **63.**)

SOLUTION.—The square of the first term is x^4, and of the second, 9. Hence, by formula **3**, letting $a = x^2$ and $b = 3$,

$$(x^2 + 3)(x^2 - 3) = x^4 - 9. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

68. Square the following:

1. $m + n$. Ans. $m^2 + 2mn + n^2$.
2. $4x + 2$. Ans. $16x^2 + 16x + 4$.
3. $3a - 5b$. Ans. $9a^2 - 30ab + 25b^2$.

Expand the following:

4. $(m + 1)(m - 1)$. Ans. $m^2 - 1$.
5. $(x^2 + y^2)(x^2 - y^2)$. Ans. $x^4 - y^4$.
6. $(4a + 4b^2)(4a - 4b^2)$. Ans. $16a^2 - 16b^4$.
7. Square $2c^2 - c + d$.

NOTE.—First separate $2c^2 - c + d$ into two terms by enclosing $c + d$ in parenthesis; then the expression becomes $2c^2 - (c - d)$, and considering this as a binomial we find the square to be $4c^4 - 4c^2(c - d) + (c - d)^2$.

$$-4c^2(c - d) = -4c^3 + 4c^2d,$$
$$(c - d)^2 = c^2 - 2cd + d^2.$$

Adding these results to $4c^4$, the final result is $4c^4 - 4c^3 + 4c^2d + c^2 - 2cd + d^2$. Ans.

DIVISION.

INTRODUCTORY.

69. When two quantities are given, and we wish to find a third quantity which, if multiplied by one of the first two, will produce the other, the process of finding this third quantity is called **division.** Thus, if the given quantities are ab and a, and we wish to find a quantity which, if multiplied by a, will give ab, we must divide ab by a; our quotient will be b, since $a \times b = ab$. Division is, therefore, the inverse of multiplication.

70. The following laws of division follow directly from the statements of Arts. **54** to **57:**

If the dividend and the divisor have like signs, the quotient will have the plus sign; if they have unlike signs, the quotient will have the minus sign.

The coefficient of the quotient is equal to the coefficient of the dividend divided by the coefficient of the divisor.

The exponent of a letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.

71. Let it be required to divide a^2 by a^2. We have to obtain a quotient, which, when multiplied by the divisor a^2, will produce the dividend a^2. The quotient is evidently 1. By Art. **70,** however, we know that $a^2 \div a^2 = a^{2-2} = a^0$. Hence, *any quantity whose exponent is 0 is equal to 1.*

From the foregoing principles, the rules for division are obtained.

DIVISION OF MONOMIALS.

72. **Rule.**—*Divide the coefficient of the dividend by the coefficient of the divisor and to the quotient annex the letters of the dividend, each with an exponent equal to its exponent in the dividend minus its exponent in the divisor, omitting those letters whose exponents become zero.*

Make the sign of the quotient plus when the dividend and divisor have like signs, and minus when they have unlike signs.

EXAMPLE.—Divide $6a^5b^4c^3$ by $-3a^2bc^3$.

SOLUTION.—The quotient of $6 \div 3$ is 2. The letters to be annexed, and their exponents, are $a^{5-2} = a^3$, and $b^{4-1} = b^3$. The c has an exponent of $3-3 = 0$, so that it becomes equal to 1, and is omitted. The sign of the quotient is minus. Hence, $6a^5b^4c^3 \div -3a^2bc^3 = -2a^3b^3$. Ans.

PROOF.— $-3a^2bc^3 \times -2a^3b^3 = 6a^5b^4c^3$.

EXAMPLE.—Divide $-10a^6b^3c^2d$ by $-2ab^3c$.

SOLUTION.— $-10a^6b^3c^2d \div -2ab^3c = 5a^{6-1}b^{3-3}c^{2-1}d = 5a^5cd$. Ans.

EXAMPLES FOR PRACTICE.

73. Divide the following:

1. $12m^2n$ by $4n$. Ans. $3m^2$.
2. $30x^6y^5bc^3$ by $-6x^5y^5c^2$. Ans. $-5xbc$.
3. $-44a^3b^3c^3$ by $-11ab^2c^3$. Ans. $4a^2b$.
4. $-100x^4y^3z^2$ by x^3y^2. Ans. $-100xyz^2$.
5. $75pq^2x^3m^4$ by $75x^3$. Ans. pq^2m^4.

DIVISION OF POLYNOMIALS.

74. When the divisor is a monomial:

Rule.—*Divide each term of the dividend by the divisor, and connect the partial quotients by their proper signs.*

EXAMPLE.—Divide $12a^2b^4 - 9ab^3 + 6a^3b^4$ by $3ab^3$.

SOLUTION.— $3ab^3) \overline{12a^2b^4 - 9ab^3 + 6a^3b^4}$

quotient $4ab - 3 + 2a^2b$. Ans.

EXAMPLES FOR PRACTICE.

75. Divide the following:

1. $64m^2n^3 - 32mn^2 + 8m^2n$ by $8mn$. Ans. $8mn^2 - 4n + m$.
2. $27x^3y^2z - 9x^3yz^2 - 333x^3y^2z^2$ by $-3x^3yz$. Ans. $-9y + 3z + 111yz$.
3. $10(x+y)^2 - 5a(x+y) + 5a^2(x+y)$ by $5(x+y)$. Ans. $2(x+y) - a + a^2$.

76. When the divisor is a polynomial:

Rule.—*Arrange both dividend and divisor according to the ascending or descending powers of some letter.*

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Subtract from the dividend the product of the divisor and this term of the quotient.

Treat the remainder as a new dividend, and proceed as before, until there is no remainder, or until the final remainder contains no term which is divisible by the first term of the divisor.

EXAMPLE.—Divide $x^4+x^3-9x^2-16x-4$ by x^2+4x+4.

SOLUTION.—

$$\begin{array}{rl} & \text{quotient.} \\ x^2+4x+4)\,x^4+x^3-9x^2-16x-4 & (x^2-3x-1. \quad \text{Ans.} \\ x^4+4x^3+4x^2 & \\ \hline -3x^3-13x^2-16x & \\ -3x^3-12x^2-12x & \\ \hline -x^2-4x-4 & \\ -x^2-4x-4 & \\ \hline 0\quad 0\quad 0 & \end{array}$$

The first term x^2 of the divisor is contained in x^4, the first term of the dividend, x^2 times; hence, x^2 is the first term of the quotient. The whole divisor multiplied by this term gives $x^4+4x^3+4x^2$ as a product, which subtracted from the dividend gives as a remainder, $-3x^3-13x^2-16x-4$. It is not necessary here to bring down the -4, since only three terms are required to contain the divisor.

The first term x^2 of the divisor is contained in $-3x^3$, the first term of the new dividend, $-3x$ times. Multiplying the divisor by this new term of the quotient, we have $-3x^3-12x^2-12x$. Subtracting this from the first remainder, we obtain $-x^2-4x-4$ for a new remainder, the -4 being brought down from the original dividend. The first term of the divisor is contained in the first term of the new remainder or dividend, -1 times. Multiplying the divisor by this, we get $-x^2-4x-4$, which subtracted from $-x^2-4x-4$, the last remainder, leaves a difference of zero. The work

ends here, since there are no more terms in the dividend to be brought down.

EXAMPLE.—Divide $9x^2y^2+x^4-4y^4-6x^3y$ by x^2+2y^2-3xy.

SOLUTION.—First arrange the dividend and divisor according to the descending powers of x.

$$
\begin{array}{l}
x^2-3xy+2y^2\,)\,x^4-6x^3y+9x^2y^2-4y^4\,(\,x^2-3xy-2y^2. \quad \text{Ans.}\\
\qquad\quad x^4-3x^3y+2x^2y^2\\
\qquad\quad\overline{\qquad\qquad\qquad\qquad}\\
\qquad\quad -3x^3y+7x^2y^2\qquad\quad -4y^4\\
\qquad\quad -3x^3y+9x^2y^2-6xy^3\\
\qquad\quad\overline{\qquad\qquad\qquad\qquad\qquad\qquad}\\
\qquad\qquad\quad -2x^2y^2+6xy^3-4y^4\\
\qquad\qquad\quad -2x^2y^2+6xy^3-4y^4\\
\qquad\qquad\quad\overline{\qquad\qquad\qquad\qquad\qquad\qquad}\\
\qquad\qquad\quad\ 0\qquad\ 0\qquad\ 0
\end{array}
$$

EXAMPLES FOR PRACTICE.

77. Divide the following:

1. $x^2-7x+12$ by $x-3$. Ans. $x-4$.
2. x^2+x-72 by $x+9$. Ans. $x-8$.
3. $2x^3-x^2+3x-9$ by $2x-3$. Ans. x^2+x+3.
4. $x^4+11x^2-12x-5x^3+6$ by $3+x^2-3x$. Ans. x^2-2x+2.
5. $x^4-6xy-9x^2-y^2$ by x^2+y+3x. Ans. x^2-3x-y.
6. x^6-1 by $x-1$. Ans. $x^5+x^4+x^3+x^2+x+1$.

FACTORING.

78. **Factoring** is the process of finding the *factors* of a quantity, that is, the quantities or numbers which will divide that quantity without a remainder.

79. Expanding $6a^2b(2b+a)$, we have $12a^2b^2+6a^3b$; hence, $6a^2b$ and $(2b+a)$ are the factors of $12a^2b^2+6a^3b$. The monomial factor $6a^2b$ may be further resolved into $3\times2\times a\times a\times b$. In solving examples in factoring, it is not customary to write out the factors of a monomial, since they are generally apparent.

That the student may be able to recognize factors without the labor of actual division, several methods of readily discovering factors are here given.

80. **Equal factors** are those whose terms have the same letters, and whose letters have the same exponents and the same signs. Thus, $5a(2y-x)$ and $5a(2y-x)$ are equal factors of $5a(2y-x)\times 5a(2y-x) = 25a^2(2y-x)^2$; but $5a(2y-x)$ and $-5a(2y-x)$ are unequal factors, since the signs of $5a$ are not the same in both expressions.

81. A product of two equal factors is a **perfect square.** Either of the equal factors of a quantity is called its square root.

82. A product of three equal factors is a **perfect cube.** Any one of the equal factors of a quantity is called its cube root.

83. In factoring, it is important to be able to easily distinguish quantities that are perfect squares and cubes, and to determine their roots. By definition, $9a^2b^2$ is a perfect square, because $3ab\times 3ab = 9a^2b^2$, and $3ab$ is its square root. Also, $8a^6$ is a perfect cube because $2a^2\times 2a^2\times 2a^2 = 8a^6$, and $2a^2$ is its cube root. In each of these cases the coefficients of the roots are multiplied together, and the exponents added, to produce a perfect power. Hence, *a quantity is a perfect square when its coefficient is a perfect square, and the exponents of all its letters can be divided by 2.* For example, $36x^{10}$, $49b^2c^4d^8$, $16a^6b^{12}$, and 1 are all perfect squares whose roots are $6x^5$, $7bc^2d^4$, $4a^3b^6$, and 1, respectively. No perfect square, however, can have a minus sign; for, let a = any quantity, $-a\times -a = a^2$, and $a\times a = a^2$. The square root of a^2 may be $-a$, or a, and a square root is often written $\pm a$, read plus or minus a.

A quantity is a perfect cube when its coefficient is a perfect cube, and the exponents of all its letters can be divided by 3. Thus, $27x^{12}$, $-64b^3c^9d^6$, $8a^{15}b^{18}$, and 1 are all perfect cubes whose roots are $3x^4$, $-4bc^3d^2$, $2a^5b^6$, and 1, respectively. The sign of the cube root is always the same as that of its cube.

CASE I.

84. *When all the terms of an expression are divisible by the same quantity, the expression may be resolved into two factors by dividing it by that quantity.*

EXAMPLE.—Factor $8m^4n^3 - 10m^3n^2y + 2m^2n^2$.

SOLUTION.—We first examine the polynomial to see if one of its terms is contained in each of the others. Beginning with the smallest coefficient, 2, we find it to be contained in each of the others. We next take the literal portion m^2n^2 of this term, and find both of its letters in each of the other terms, with an equal or higher exponent. Dividing through by $2m^2n^2$, we have for our quotient $4m^2n - 5my + 1$. Hence, we have

$$8m^4n^3 - 10m^3n^2y + 2m^2n^2 = 2m^2n^2(4m^2n - 5my + 1). \quad \text{Ans.}$$

EXAMPLE.—Ascertain if $12ab^2c^3 - 18a^3c^2y + 24a^2c^4 - 36a^2bc^3y^2$ has a monomial factor.

SOLUTION.—By inspection we find that the smallest coefficient is not contained in the other three without a remainder. This coefficient, 12, is the product of $3 \times 2 \times 2$. Of these, both 2 and 3 will divide all the coefficients once. Therefore, $2 \times 3 = 6$ is the largest numerical factor. The letters a and c are contained in all the terms; the smallest exponent of a is 1 and of c is 2. The monomial factor is evidently $6ac^2$. Dividing the polynomial by $6ac^2$, the quotient is $2b^2c - 3a^2y + 4ac^2 - 6abc^3y^2$; the factors are $6ac^2$ and $2b^2c - 3a^2y + 4ac^2 - 6abc^3y^2$. Ans.

EXAMPLES FOR PRACTICE.

85. Factor the following expressions:

1. $a^4 + ax$. Ans. $a(a^3 + x)$.
2. $12a^5 - 2a^3 + 4a^4$. Ans. $2a^3(6a^2 - 1 + 2a)$.
3. $30m^4n^2 - 6n^3$. Ans. $6n^2(5m^4 - n)$.
4. $16x^2y^3 - 8x^5 + 8$. Ans. $8(2x^2y^3 - x^5 + 1)$.
5. $4x^3y - 12x^2y^2 + 8xy^3$. Ans. $4xy(x^2 - 3xy + 2y^2)$.
6. $49a^2b^3c^4 - 63a^3b^2c^4 + 7a^4b^2c^3$. Ans. $7a^2b^2c^3(7bc - 9ac + a^2)$.

CASE II.

86. *To factor a trinomial which is a perfect square:*

Any trinomial is a perfect square when the first and the last term are perfect squares and positive, and the second term is twice the product of their square roots.

Thus, let a and b represent any two quantities whatever, and we have the general forms of the square as follows:

$$a^2+2ab+b^2 = (a+b)(a+b) = (a+b)^2. \qquad (4.)$$

$$a^2-2ab+b^2 = (a-b)(a-b) = (a-b)^2. \qquad (5.)$$

These, it will be seen, are simply the inverse of formulas **1** and **2**, Art. **65**. The sign of the second term of the square always determines the sign of the second term of the root, b in this particular case.

87. Since a may represent one quantity and b any other quantity, it is evident that *any* trinomial having the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$ is a perfect square.

Rule.—*Extract the square roots of the first and the last term of the trinomial, and connect the results by the sign of the second term.*

EXAMPLE.—Factor $x^2+2xy+y^2$.

SOLUTION.—We first see if the trinomial has the form stated in Art. **86**. The first and the last term we see to be perfect squares, and their roots to be x and y. The second term is also twice the product of the roots x and y, and, since it has the plus sign, the binomial root must be $x+y$. Hence, we have a square of the form $a^2+2ab+b^2$, and

$$x^2+2xy+y^2 = (x+y)(x+y) = (x+y)^2. \quad \text{Ans.}$$

EXAMPLE.—Factor $16m^4+9n^6-24m^2n^3$.

SOLUTION.—The first term of the expression is a perfect square, but the last term is not. Inspecting the second term, we find it to be the square of $3n^3$, and the third term to be twice the product of $3n^3$ and the square root, $4m^2$, of the first term. Arranging the trinomial so that the first and the last term are perfect squares, we get $16m^4-24m^2n^3+9n^6$ (a square of the form $a^2-2ab+b^2$), and we have $16m^4+9n^6-24m^2n^3 = 16m^4-24m^2n^3+9n^6 = (4m^2-3n^3)(4m^2-3n^3) = (4m^2-3n^3)^2$. Ans.

EXAMPLE.—Factor $4x^2+x^2y^2+2x^2y$.

SOLUTION.—Arranging the trinomial so that the first and the last term are perfect squares, we have $4x^2+2x^2y+x^2y^2$. Now, although the first and the last term are perfect squares with roots $2x$ and xy, respectively, the second term is only equal to the product of the roots; hence, the trinomial is *not* a perfect square, and can only be factored by Case I. Each term contains x^2, and we have $4x^2+x^2y^2+2x^2y = x^2(4+y^2+2y)$. Ans.

88. *Should two of the terms of a trinomial be perfect squares, and have like signs, and the other term be twice the product of their roots, the trinomial is a perfect square.*

Compare this statement with Art. **86.** Thus, $2ab-a^2-b^2$, if divided by -1, becomes $-2ab+a^2+b^2 = a^2-2ab+b^2$; hence, $2ab-a^2-b^2 = -(a^2-2ab+b^2) = -(a-b)^2$.

EXAMPLE.—Factor $4pq-4p^2-q^2$.

SOLUTION.—Dividing first by -1 we have $-4pq+4p^2+q^2 = 4p^2-4pq+q^2 = (2p-q)^2$. Hence, $4pq-4p^2-q^2 = -(4p^2-4pq+q^2) = -(2p-q)^2$. Ans.

EXAMPLE.—Factor $16r^2s^2+16r^4+4s^4$.

SOLUTION.—The expression contains three squares, but, by careful inspection, we see that the first term is also twice the product of the square roots of the other two. Thus,

$16r^2s^2+16r^4+4s^4 = 16r^4+16r^2s^2+4s^4 = (4r^2+2s^2)^2$. Ans.

EXAMPLES FOR PRACTICE.

89. Factor the following trinomials:

1. $x^2-16x+64$. Ans. $(x-8)^2$.
2. n^6-26n^3+169. Ans. $(n^3-13)^2$.
3. $25x^2+70xyz+49y^2z^2$. Ans. $(5x+7yz)^2$.
4. $16c^2+b^2-8bc$. Ans. $(4c-b)^2$.
5. $2mx-m^2-x^2$. Ans. $-(m-x)^2$.
6. $a^2b^4c^6-2ab^2c^3+1$. Ans. $(ab^2c^3-1)^2$.

CASE III.

90. *To factor an expression which is the difference between two perfect squares:*

This case is the inverse of formula **3**, Art. **65**, and may be expressed by the formula

$$a^2-b^2 = (a+b)(a-b). \qquad (6.)$$

91. Since a may represent one quantity and b any other quantity, it is evident from formula **6** that any expression which is the difference between two perfect squares may be factored by the following

Rule.—*Extract the square roots of the first and the last term. Write the first root plus the second for one factor, and the first root minus the second for the other.*

EXAMPLE.—Factor $9x^8y^6 - 4$.

SOLUTION.—The square roots of the first and the last term are $3x^4y^3$ and 2. The sum of these roots is $3x^4y^3 + 2$, and the second subtracted from the first is $3x^4y^3 - 2$. Hence, by formula **6**, letting $a = 3x^4y^3$ and $b = 2$,

$$9x^8y^6 - 4 = (3x^4y^3 + 2)(3x^4y^3 - 2). \quad \text{Ans.}$$

EXAMPLE.—Factor $(a+b)^2 - m^2n^2$.

SOLUTION.—The square roots of the first and the last term are $a+b$ and mn. The sum of these roots is $a+b+mn$, and the second subtracted from the first is $a+b-mn$. Hence, by formula **6**, letting $a = a+b$ and $b = mn$,

$$(a+b)^2 - m^2n^2 = (a+b+mn)(a+b-mn). \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

92. Factor the following expressions:

1. $a^2 - 16$. Ans. $(a+4)(a-4)$.
2. $a^2 - 49c^8$. Ans. $(a+7c^4)(a-7c^4)$.
3. $81x^6y^4 - 1$. Ans. $(9x^3y^2+1)(9x^3y^2-1)$.
4. $(ax+by)^2 - 1$. Ans. $(ax+by+1)(ax+by-1)$.
5. $25x^4y^2 - (bx+1)^2$. Ans. $[5x^2y + (bx+1)][5x^2y - (bx+1)]$ $= (5x^2y + bx + 1)(5x^2y - bx - 1)$.
6. $1 - 169x^2y^4z^6$. Ans. $(1+13xy^2z^3)(1-13xy^2z^3)$.

93. In example 5, the expression $(bx+1)^2$ should be regarded as a single term; in fact, any number of terms may be regarded as a single term by enclosing them in parenthesis and operating on them as though they were a single letter.

94. When solving any examples requiring the application of the rule in Art. **91**, first ascertain if the numerical coefficients of the two terms are perfect squares; if not, there is no use of examining further.

FRACTIONS.

DEFINITIONS.

95. A **fraction,** in algebra, is considered as an expression indicating division. The sign ÷ is seldom used, it being more convenient to write the dividend, or quantity to be divided, above a horizontal line, with the divisor below it, in the form of a fraction.

Thus the fraction $\frac{a+b}{c-d}$ means that $a+b$ is to be divided by $c-d$, and is the same as $(a+b) \div (c-d)$. It is read "$a+b$ divided by $c-d$" or "$a+b$ *over* $c-d$." All fractions are read in this way in algebra, except simple numerical fractions, as $\frac{1}{2}$, $\frac{2}{3}\frac{5}{2}$, etc., which are read as in arithmetic.

96. The quantities above and below the line are called the **numerator** and the **denominator,** respectively, as in the case of numerical fractions. They are known as the **terms** of a fraction.

97. Since dividing any quantity by 1 does not change its value, we may write any quantity as a fraction by making the quantity itself the numerator, and 1 the denominator. Thus, $7x^2y$ may be written $\frac{7x^2y}{1}$, and not be altered in value.

98. A **reciprocal** of a quantity is 1 divided by that quantity. It is not necessarily written as a fraction, but when so written has the quantity for the denominator, and 1 for the numerator. Thus, the reciprocal of 5 is $\frac{1}{5}$, but the reciprocal of $\frac{1}{5}$ is $\frac{5}{1} = 5$; the reciprocal of $x^2+y^2 = \frac{1}{x^2+y^2}$; and the reciprocal of $\frac{2}{x^2+y^2} = \frac{x^2+y^2}{2}$. Hence, the reciprocal of a fraction may be obtained by inverting the fraction.

99. The **three signs of a fraction** are: the sign before the dividing line, which affects the entire fraction; the sign of the numerator; and the sign of the denominator. When any one of these signs is omitted, it is understood to be plus.

Any two signs of a fraction may be changed without altering its value, but if any one, or all three, be changed, the value of the fraction will be changed from + to − or from − to +.

When either the numerator or the denominator has more than one term, it should be enclosed in a parenthesis when performing operations affecting it as a whole. The parenthesis may be removed after the operations are completed.

Take the fraction $-\frac{a-b}{c-d}$; placing numerator and denominator in parentheses, we have $-\frac{(a-b)}{(c-d)}$. The signs of the numerator and denominator are each + and that of the fraction −.

Let the quotient of $a-b \div c-d = q$; then:

$$-\frac{+(a-b)}{+(c-d)} = -(+q) = -q.$$

$$-\frac{-(a-b)}{-(c-d)} = -(+q) = -q.$$

$$-\frac{+(a-b)}{-(c-d)} = -(-q) = +q.$$

$$-\frac{-(a-b)}{+(c-d)} = -(-q) = +q.$$

$$+\frac{+(a-b)}{+(c-d)} = +(+q) = +q.$$

$$+\frac{-(a-b)}{-(c-d)} = +(+q) = +q.$$

$$+\frac{+(a-b)}{-(c-d)} = +(-q) = -q.$$

$$+\frac{-(a-b)}{+(c-d)} = +(-q) = -q.$$

REDUCTION OF FRACTIONS.

100. To *reduce* a fraction is to change its form without changing its value. Thus, $\frac{10x}{5}$ and $\frac{20x}{10}$ have different forms, but like values, since $10x \div 5$ and $20x \div 10$ are each equal to $2x$.

The terms of a fraction may both be multiplied, or may both be divided by the same quantity without changing their value.

101. To reduce a fraction to its simplest form:

Rule.—*Resolve each term into its factors, and cancel factors which appear in both.*

102. *In performing all operations on fractions, the student must learn to use a polynomial factor as a single quantity, like a monomial factor.*

This is illustrated in the following examples, where there are polynomial factors in both numerator and denominator that can be canceled.

EXAMPLE.—Reduce $\frac{x^2+2xy+y^2}{x^2-y^2}$ to its simplest form.

SOLUTION.—Factoring both numerator and denominator,

$$\frac{x^2+2xy+y^2}{x^2-y^2} = \frac{(x+y)(x+y)}{(x+y)(x-y)}.$$

Canceling the common factor $x+y$ from both, gives as the result,

$$\frac{\cancel{(x+y)}(x+y)}{\cancel{(x+y)}(x-y)} = \frac{x+y}{x-y}. \quad \text{Ans.}$$

EXAMPLE.—Reduce $\frac{3x^5-6x^4y}{6x^2y^2-12xy^3}$ to its simplest form.

SOLUTION.— $\frac{3x^5-6x^4y}{6x^2y^2-12xy^3} = \frac{3x^4(x-2y)}{6xy^2(x-2y)}$, when factored.

Canceling the common factors, we have as the result,

$$\frac{\overset{x^3}{\cancel{3x^4}}\cancel{(x-2y)}}{\underset{2}{\cancel{6xy^2}}\cancel{(x-2y)}} = \frac{x^3}{2y^2}. \quad \text{Ans.}$$

103. Sometimes the whole numerator is contained in the denominator, or the denominator in the numerator. The numerator or denominator will then reduce to the number 1.

EXAMPLE.—Reduce $\frac{b+3c^2}{2b^2+6bc^2}$ to its simplest form.

SOLUTION.— $\frac{b+3c^2}{2b^2+6bc^2} = \frac{\cancel{b+3c^2}}{2b\cancel{(b+3c^2)}} = \frac{1}{2b}$. Ans.

EXAMPLE.—Reduce $\frac{x^6-1}{x^3-1}$ to its simplest form.

SOLUTION.— $\frac{x^6-1}{x^3-1} = \frac{(x^3+1)(x^3-1)}{x^3-1} = \frac{x^3+1}{1} = x^3+1$. Ans.

(Art. **97.**)

104. From the last example it will be seen that division may sometimes be performed by cancelation. Thus, $\frac{x^6-1}{x^3-1}$ means $(x^6-1) \div (x^3-1)$, and the divisor x^3-1 canceled from the dividend x^6-1 gives the quotient x^3+1.

A factor must be common to each term of the numerator and to each term of the denominator in order to be canceled. Thus, the factor x cannot be canceled from $\frac{3ax}{x+4m}$ because it is not common to both terms of the denominator.

EXAMPLES FOR PRACTICE.

105. Reduce the following to their simplest form

1. $\frac{3a+3b}{a^2-b^2}$. Ans. $\frac{3}{a-b}$.
2. $\frac{x^4-y^4}{x^2-y^2}$. Ans. x^2+y^2.
3. $\frac{54a^3b^5c^2}{72a^2b^2c}$. Ans. $\frac{3ab^3c}{4}$.
4. $\frac{12a^2x^3}{36a^3x^5}$. Ans. $\frac{1}{3ax^2}$.
5. $\frac{n^3-2n^2}{n^2-4n+4}$. Ans. $\frac{n^2}{n-2}$.

106. When fractions are to be added or subtracted, it is necessary to so reduce them that all the denominators will be alike. This is called **reducing them to a common denominator.**

107. To reduce fractions to a common denominator:

Rule.—*Resolve each denominator into its factors.*

Take each factor as many times as it occurs in any one denominator, and find the product of these factors.

Divide this product by each of the denominators. Multiply the corresponding numerators by these quotients, for new numerators. Write each new numerator with the common denominator beneath it.

EXAMPLE.—Reduce $\frac{7a}{x+y}$, $\frac{3ab}{x^2-y^2}$, and $\frac{2b}{(x+y)^2}$ to a common denominator.

SOLUTION.—Factoring the denominators, we have $x+y$ not factorable. $x^2-y^2 = (x+y)(x-y)$, and $(x+y)^2 = (x+y)(x+y)$. Now we have two separate factors, $x+y$ and $x-y$, of which $x+y$ occurs twice in $(x+y)^2$. Hence, our common denominator is $(x+y)(x+y)(x-y) = (x+y)^2(x-y)$. Dividing this product by $x+y$ we have $(x+y)(x-y) = x^2-y^2$ as our quotient. Hence, our first new numerator is $7a(x^2-y^2)$ and the new fraction is $\frac{7a(x^2-y^2)}{(x+y)^2(x-y)}$. Similarly, $\frac{3ab}{x^2-y^2}$ becomes $\frac{3ab(x+y)}{(x+y)^2(x-y)}$, and $\frac{2b}{(x+y)^2}$ becomes $\frac{2b(x-y)}{(x+y)^2(x-y)}$.

The student should note that this denominator can be written in several different ways, and he should not become confused if his work does not always agree with the answer. Besides $(x+y)(x+y)(x-y)$ and $(x+y)^2(x-y)$, it may be written $(x^2-y^2)(x+y)$, $(x^2+2xy+y^2)(x-y)$, or $x^3+x^2y-xy^2-y^3$. These five expressions have exactly the same value. The student should prove this statement by substituting numbers for x and y.

EXAMPLES FOR PRACTICE.

108. Reduce the following to a common denominator:

1. $\frac{3yz}{2x}$, $\frac{4xz}{3y}$, and $\frac{5xy}{4z}$. Ans. $\frac{18y^2z^2}{12xyz}$, $\frac{16x^2z^2}{12xyz}$, and $\frac{15x^2y^2}{12xyz}$.

2. $\frac{x^2y}{10}$, $\frac{xyz}{15}$, and $\frac{7yz^2}{30}$. Ans. $\frac{3x^2y}{30}$, $\frac{2xyz}{30}$, and $\frac{7yz^2}{30}$.

3. $\frac{2}{a^3x^3}$, $\frac{3}{ax^3}$, and $\frac{4}{a^2x}$. Ans. $\frac{2}{a^3x^3}$, $\frac{3a^2}{a^3x^3}$, and $\frac{4ax^2}{a^3x^3}$.

4. $\frac{m+n}{m-n}$, and $\frac{m-n}{m+n}$. Ans. $\frac{m^2+2mn+n^2}{m^2-n^2}$, and $\frac{m^2-2mn+n^2}{m^2-n^2}$.

5. $\frac{2}{x}$, $\frac{3}{2x-1}$, and $\frac{2x-1}{4x^2-1}$.

Ans. $\frac{2(4x^2-1)}{x(4x^2-1)}$, $\frac{3x(2x+1)}{x(4x^2-1)}$, and $\frac{x(2x-1)}{x(4x^2-1)}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

109. To add or subtract fractions:

Rule.—*Reduce the fractions, if necessary, to a common denominator. Add or subtract the numerators, and write the result over the common denominator.*

EXAMPLE.—Find the sum of $\frac{2a-b}{5}$ and $\frac{a+b}{4}$.

SOLUTION.— $\frac{2a-b}{5}$ and $\frac{a+b}{4}$, reduced to a common denominator, become $\frac{4(2a-b)}{20}$ and $\frac{5(a+b)}{20}$, which are equal, respectively, to $\frac{8a-4b}{20}$ and $\frac{5a+5b}{20}$. Adding the numerators, we have $8a-4b+5a+5b = 13a+b$. The result written over the common denominator gives as the sum, $\frac{13a+b}{20}$. The work is written as follows:

$$\frac{2a-b}{5}+\frac{a+b}{4} = \frac{8a-4b}{20}+\frac{5a+5b}{20}$$
$$= \frac{8a-4b+5a+5b}{20} = \frac{13a+b}{20}. \quad \text{Ans.}$$

EXAMPLE.—Subtract $\frac{6b-2}{3b}$ from $\frac{4a-1}{2a}$.

SOLUTION.—Reducing the fractions to a common denominator, $\frac{4a-1}{2a} - \frac{6b-2}{3b} = \frac{12ab-3b}{6ab} - \frac{12ab-4a}{6ab}$. Subtracting the second numerator from the first, and writing the result over the common denominator, we have $\frac{12ab-3b}{6ab} - \frac{12ab-4a}{6ab} = \frac{(12ab-3b)-(12ab-4a)}{6ab}$ $= \frac{12ab-3b-12ab+4a}{6ab}$, with the parentheses removed. Combining like terms in the numerator gives as the result $\frac{4a-3b}{6ab}$. Ans.

110. *If, as in the example just given, the numerator of the fraction to be subtracted has more than one term, care must be taken to change the sign of every term before combining.* It will usually be convenient to enclose the whole numerator in a parenthesis before combining. The parenthesis may then be removed by the rules of Arts. **50** and **51.**

EXAMPLE.—Simplify $\frac{x^3}{x-1}-\frac{x^2}{x+1}-\frac{x}{x-1}+\frac{1}{x+1}$.

SOLUTION.—Reducing to the common denominator x^2-1,

$$\frac{x^3}{x-1}-\frac{x^2}{x+1}-\frac{x}{x-1}+\frac{1}{x+1}=\frac{x^4+x^3}{x^2-1}-\frac{x^3-x^2}{x^2-1}-\frac{x^2+x}{x^2-1}+\frac{x-1}{x^2-1}.$$

Adding or subtracting the numerators as required, we have

$$\frac{(x^4+x^3)-(x^3-x^2)-(x^2+x)+(x-1)}{x^2-1},$$

which, with the parentheses removed, equals

$$\frac{x^4+x^3-x^3+x^2-x^2-x+x-1}{x^2-1}.$$

Combining like terms we have as the result

$$\frac{x^4-1}{x^2-1}=x^2+1. \quad \text{Ans.}$$

EXAMPLE.—Simplify $\frac{1}{(x-2)^2}+\frac{1}{2-x}$.

SOLUTION.—If the denominator of the second fraction were written $x-2$ instead of $2-x$, $(x-2)^2$ would be the common denominator. By Art. **99**, the signs of the denominator and the sign before the fraction $\frac{1}{2-x}$ may be changed, giving $-\frac{1}{-2+x}=-\frac{1}{x-2}$. (Art. **24.**) Hence, we have $\frac{1}{(x-2)^2}+\frac{1}{2-x}=\frac{1}{(x-2)^2}-\frac{1}{x-2}$, which, when reduced to a common denominator, is equal to

$$\frac{1}{(x-2)^2}-\frac{x-2}{(x-2)^2}=\frac{1-(x-2)}{(x-2)^2}=\frac{1-x+2}{(x-2)^2}=\frac{3-x}{(x-2)^2}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

111. Simplify the following:

1. $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}$. Ans. $\frac{47x}{60}$.

2. $\frac{4x-3}{5}+\frac{7x+1}{3}+\frac{3x}{2}$. Ans. $\frac{139x-8}{30}$.

3. $\frac{1}{x-y}-\frac{1}{x^2-y^2}$. Ans. $\frac{x+y-1}{x^2-y^2}$.

4. $\frac{a^2+b^2}{2}-\frac{(a+b)^2}{4}$. Ans. $\frac{2(a^2+b^2)-(a^2+2ab+b^2)}{4}=\frac{(a-b)^2}{4}$, after removing parentheses and combining.

5. $\frac{a^2}{a^2-1}+\frac{a}{a-1}-\frac{a}{a+1}$. Ans. $\frac{a^2+2a}{a^2-1}$.

6. $\frac{4m^2+1}{4m^2}-\frac{3m-1}{12m^3}+\frac{1-12n}{12n}$. Ans. $\frac{n+m^3}{12m^3n}$.

7. $\frac{y}{(x+y)^2}+\frac{y}{x^2-y^2}-\frac{1}{x+y}$.

Ans. $\frac{y(x-y)+y(x+y)-(x^2-y^2)}{(x+y)^2(x-y)}=\frac{2xy-x^2+y^2}{x^3+x^2y-xy^2-y^3}$.

8. $\frac{x}{x+1}+\frac{x}{1-x}+\frac{3x}{x^2-1}$. Ans. $\frac{x}{x^2-1}$.

MULTIPLICATION OF FRACTIONS.

112. Multiplication, in fractions, is the process of finding a fractional part of a fraction. Thus, $\frac{3}{4}\times\frac{1}{2}$ means $\frac{1}{2}$ of $\frac{3}{4}$. One-half of $\frac{3}{4}$ inch, for example, is $\frac{3}{8}$ inch; $\frac{2}{3}$ of $\frac{3}{4}$ inch is $\frac{1}{2}$, or $\frac{6}{12}$, inch. The result in each case is the same as that which would be obtained by finding the product of the numerators and writing it over the product of the denominators.

113. Hence, to multiply fractions:

Rule.—*Multiply the numerators together for the numerator of the product, and the denominators together for the denominator of the product.*

114. Any number of fractions may be multiplied together. The operation may be very much shortened by resolving the terms of the fractions into their factors, and canceling. The product should be reduced to its simplest form.

EXAMPLE.—Find the product of $\frac{6a^2}{5}$, $\frac{2ab}{3c}$, and $\frac{2ac}{b^2}$.

SOLUTION.—The product of the numerators is $6a^2\times 2ab\times 2ac=24a^4bc$, and of the denominators, $5\times 3c\times b^2=15b^2c$. Writing $24a^4bc$ over $15b^2c$, we have for the product, $\frac{24a^4bc}{15b^2c}=\frac{8a^4}{5b}$, when reduced to its lowest terms. The work is written as follows:

$$\frac{\overset{2}{\cancel{6}a^{\cancel{2}}}}{5}\times\frac{2a\cancel{b}}{\cancel{3}\cancel{c}}\times\frac{2a\cancel{c}}{\cancel{b^2}\,\cancel{b}}=\frac{8a^4}{5b}.$$ Ans.

EXAMPLE.—Find the product of $\frac{x^2+2x}{(x-1)^2}$, x^2-1, and $\frac{x^2-4x+4}{x^2-4}$.

SOLUTION.—First make x^2-1 a fraction by writing 1 for its denominator, thus, $\frac{x^2-1}{1}$; then, factoring both terms of each fraction, we have

$$\frac{x^2+2x}{(x-1)^2}\times\frac{x^2-1}{1}\times\frac{x^2-4x+4}{x^2-4} =$$

$$\frac{x(x+2)(x+1)(x-1)(x-2)(x-2)}{(x-1)(x-1)(x-2)(x+2)} = \frac{x(x+1)(x-2)}{x-1}. \quad \text{Ans.}$$

EXAMPLE.—Find the product of $\frac{1}{a^3}-\frac{4c^2}{a}$, and $\frac{a^3}{1+2ac}$.

SOLUTION.—Performing the subtraction, $\frac{1}{a^3}-\frac{4c^2}{a} = \frac{1-4a^2c^2}{a^3}$.

Multiplying, $\frac{1-4a^2c^2}{a^3}\times\frac{a^3}{1+2ac} = \frac{1-2ac}{(1+2ac)(1-2ac)}\frac{a^3}{a^3} = 1-2ac.$ Ans.

EXAMPLES FOR PRACTICE.

115. Multiply the following:

1. $\frac{3a^2bc}{5abc^2}$ by $\frac{10ab^2c}{3abc}$. Ans. $\frac{2ab}{c}$.

2. $\frac{5x^2y}{7x}$ by $21xy$. Ans. $15x^2y^2$.

Find the product of:

3. $\frac{3x^2y}{4xz^2}$, $\frac{5y^2z}{6xy}$, and $\frac{-12x^2}{2xy^2}$. Ans. $-\frac{15x}{4z}$.

4. $\frac{x^2-y^2}{c^2-d^2}$, $\frac{c-d}{(x+y)^2}$, and $\frac{x^3+y^3}{x-y}$. Ans. $\frac{x^2-xy+y^2}{c^2+cd+d^2}$.

5. $\frac{4y}{x}-\frac{16}{xy}$, and $\frac{1}{2y+4}$. Ans. $\frac{2y-4}{xy}$.

6. $\frac{a+b}{2}+\frac{a-b}{4}$, and $\frac{4}{9a^2+6ab+b^2}$. Ans. $\frac{1}{3a+b}$.

DIVISION OF FRACTIONS.

116. Division, in fractions, is the reverse of multiplication, and is the process we use when, given one of two fractions and their product, we are required to find the other. For example, we are required to divide $\frac{a}{4}$ by $\frac{1}{2}$. We wish

to find such a fraction that, multiplied by $\frac{1}{2}$, will give $\frac{a}{4}$. This fraction is $\frac{a}{2}$, for $\frac{a}{2}\times\frac{1}{2}=\frac{a}{4}$. Also $\frac{x}{5}\div\frac{x}{7}=\frac{7}{5}$, since $\frac{7}{5}\times\frac{x}{7}=\frac{x}{5}$. If, in this case, we had inverted the divisor and multiplied, we should have had $\frac{x}{5}\times\frac{7}{x}=\frac{7}{5}$.

117. Hence, to divide by a fraction:

Rule.—*Invert the divisor, and proceed as in multiplication.*

EXAMPLE.—Divide $\frac{3a^2b}{5x^3y}$ by $\frac{9ab^3}{10x^4y^2}$.

SOLUTION.—The divisor inverted $=\frac{10x^4y^2}{9ab^3}$.

Hence, $\frac{3a^2b}{5x^3y}\div\frac{9ab^3}{10x^4y^2}=\frac{3a^2b}{5x^3y}\times\frac{10x^4y^2}{9ab^3}=\frac{\cancel{3}\times\cancel{10}\overset{2a}{\cancel{a^2}}\cancel{b}\overset{x}{\cancel{x^3}}\overset{y}{\cancel{y^2}}}{\cancel{5}\times\cancel{9}\underset{3}{\cancel{a}}\cancel{b^3}\cancel{x^3}\cancel{y}}=\frac{2axy}{3b^2}$. Ans.

EXAMPLE.—Divide x^2+2x+1 by $\frac{x+1}{x-1}$.

SOLUTION.—By Art. **97**, $(x^2+2x+1)\div\frac{x+1}{x-1}=$

$$\frac{x^2+2x+1}{1}\times\frac{x-1}{x+1}=\frac{\cancel{(x+1)}(x+1)(x-1)}{\cancel{x+1}}=x^2-1.$$ Ans.

EXAMPLES FOR PRACTICE.

118. Divide the following:

1. $\frac{9x^3-3x^4}{24}$ by $\frac{3x}{8}$. Ans. $\frac{3x^2-x^3}{3}$.
2. $\frac{ab-bx}{a+z}$ by $\frac{ac-cx}{a+z}$. Ans. $\frac{b}{c}$.
3. $\frac{1-8b^2+16b^4}{1+2b}$ by $\frac{1-4b^2}{3a}$. Ans. $3a(1-2b)$.
4. $6a^2cd-6abcd$ by $\frac{6acd}{a^2+ab+b^2}$. Ans. a^3-b^3.

MIXED QUANTITIES AND COMPLEX FRACTIONS.

119. An **integral expression** is one containing neither fractions nor negative exponents. The expression a^2+2ab is integral, but the expressions $a^2+\frac{1}{2ab}$, $2a^{-2}$, $\frac{3}{a^2+b}$ are not. The expression $2a^{-2}$ is only another way of writing $\frac{2}{a^2}$.

The use of negative exponents will be explained in subsequent paragraphs.

120. The **integral part** of an expression is that part which, if taken by itself, would be an integral expression.

121. A **mixed quantity** is an expression containing both integral and fractional parts, as $2a^2-\frac{c+d}{4}$. Considering the integral part, $2a^2$, as a fraction with a denominator 1 (Art. **97**), a mixed quantity becomes simply the indicated addition or subtraction of two fractions; thus,

$$2a^2-\frac{c+d}{4}=\frac{2a^2}{1}-\frac{c+d}{4}.$$

122. A fraction may be reduced to either an entire or mixed quantity by dividing the numerator by the denominator, provided the division be possible. It frequently happens that by performing the indicated division, the fraction will be reduced to a simpler form. The case of reducing a fraction to an entire quantity was taken up in Art. **103.**

EXAMPLE.—Simplify $\frac{4x^2+12x-1}{2x+3}$.

SOLUTION.—Performing the indicated division,

$$\begin{array}{r} 2x+3\,)\,4x^2+12x-1\,(\,2x+3-\frac{10}{2x+3}. \quad \text{Ans.} \\ 4x^2+6x \\ \hline 6x-1 \\ 6x+9 \\ \hline -10 \end{array}$$

123. Mixed quantities are frequently more convenient to handle as fractions.

To reduce a mixed quantity to a fraction:

Rule.—*Write the integral part with a denominator 1, and perform the indicated addition or subtraction.*

EXAMPLE.—Reduce $x^2+xy+y^2-\frac{b}{x-y}$ to a fraction.

SOLUTION.— $x^2+xy+y^2-\frac{b}{x-y}=\frac{x^2+xy+y^2}{1}-\frac{b}{x-y}$; subtracting the second fraction from the first gives

$$\frac{(x^2+xy+y^2)(x-y)-b}{x-y}=\frac{x^3-y^3-b}{x-y}.$$ Ans.

EXAMPLES FOR PRACTICE.

124. Solve the following:

1. Reduce $\frac{a^2c+b^2}{c}$ to a mixed quantity. Ans. $a^2+\frac{b^2}{c}$.

2. Simplify $\frac{x^2+4xy+5y^2-3x}{x+2y}$. Ans. $x+2y-3+\frac{y^2+6y}{x+2y}$.

3. Reduce $x+3-\frac{7x+3}{2x+1}$ to a fraction. Ans. $\frac{2x^2}{2x+1}$.

4. From $3a+\frac{a+b}{d}$ subtract $a-\frac{a-b}{d}$. Ans. $2a+\frac{2a}{d}=\frac{2a(d+1)}{d}$.

5. Divide $m+n-\frac{2n}{m-n}$ by $m-n-\frac{2n}{m+n}$. Ans. $\frac{m+n}{m-n}$.

SUGGESTION.—First reduce the mixed quantities to fractions.

125. A **complex fraction** is one which contains fractions in one or both of its terms. Thus, $\frac{a+\frac{x}{y}}{a+x}$, $\frac{a-b}{\frac{x}{y}}$, and $\frac{\frac{a}{b}}{\frac{c}{d}}$ are complex fractions.

126. Complex fractions can be reduced by performing the indicated division; thus, $\frac{\frac{5}{8}}{\frac{3}{4}}=\frac{5}{8}\div\frac{3}{4}=\frac{5}{8}\times\frac{4}{3}=\frac{5}{6}$. A much simpler way is to multiply both terms by the least

common denominator of the fractions contained. Thus, $\frac{\frac{5}{8}\times 8}{\frac{3}{4}\times 8} = \frac{5}{6}$.

127. Hence, to simplify a complex fraction:

Rule.—*Multiply both terms by the common denominator of the fractional parts.*

EXAMPLE.—Simplify $\dfrac{\frac{x}{y}-\frac{y}{x}}{\frac{1}{y}-\frac{1}{x}}$.

SOLUTION.—The common denominator of the fractional parts is xy. Multiplying each term by this, we have

$$\frac{\frac{x}{y}\times xy-\frac{y}{x}\times xy}{\frac{1}{y}\times xy-\frac{1}{x}\times xy} = \frac{x^2-y^2}{x-y} = x+y. \quad \text{Ans.}$$

The multiplication can frequently be performed mentally, without writing the common denominator, at the same time canceling common factors.

EXAMPLE.—Simplify $\dfrac{1}{1+\dfrac{a}{1+a+\dfrac{2a^2}{1-a}}}$.

SOLUTION.—This is the case of a complex fraction in which the denominator is itself a complex fraction.

First, consider the part $\dfrac{a}{1+a+\dfrac{2a^2}{1-a}}$.

Multiplying both terms by $1-a$, we have

$$\frac{a(1-a)}{(1+a)(1-a)+2a^2} = \frac{a-a^2}{1-a^2+2a^2} = \frac{a-a^2}{1+a^2}.$$

The fraction thus becomes $\dfrac{1}{1+\dfrac{a-a^2}{1+a^2}}$.

Multiplying both terms by $1+a^2$, the common denominator, we have

$$\frac{1+a^2}{1+a^2+a-a^2} = \frac{1+a^2}{1+a}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

128. Simplify the following:

1. $\dfrac{\frac{3ac^2}{16}}{\frac{24}{}}$. Ans. $\dfrac{ac^2}{128}$.

2. $\dfrac{1+\frac{a}{c}}{c-\frac{a^2}{c}}$. Ans. $\dfrac{1}{c-a}$.

3. $\dfrac{2\frac{7}{8}}{8-2x+\frac{x^2}{8}}$. Ans. $\dfrac{23}{(8-x)^2}$.

SUGGESTION.— $2\frac{7}{8}$ means $2+\frac{7}{8}$. Hence, for the numerator, multiply 2 by the least common denominator 8, and add 7.

4. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$. Ans. $\dfrac{4}{3x+3}$.

THEORY OF EXPONENTS.

129. An exponent has already been defined in Art. **12.** In addition to positive integral exponents, as there defined, we frequently have to deal with negative and with fractional exponents. Thus, in the quantities a^3, a^{-3}, and $a^{\frac{2}{3}}$, we have the positive integral exponent 3, the negative exponent -3, and the fractional exponent $\frac{2}{3}$. The rules for positive and integral exponents apply also to negative and fractional exponents.

130. Since letters may represent numbers, they may be used for exponents, the same as figures. Thus, a^n means that a is to be taken as many times as a factor as there are units in n, or $a \times a \times a$, etc. to n factors. Such exponents are called **literal exponents.** Fractional, negative, and literal exponents are all read by using the word exponent. Thus, $a^{\frac{2}{3}}$, a^{-4}, $a^{\frac{n}{m}}$, and a^n are read, "a, exponent $\frac{2}{3}$," "a, exponent minus 4," "a, exponent n over m," and "a, exponent n," respectively.

131. A **fractional exponent** is an expression of a root or of a power and a root combined. The numerator of the fraction denotes the power, and the denominator denotes the root. Thus, $\sqrt[3]{a}$ may be written $a^{\frac{1}{3}}$. Suppose we wish to find the cube root of a^6; instead of writing it $\sqrt[3]{a^6} = a^2$, we may write it $a^{\frac{6}{3}} = a^2$. Hence,

The numerator of a fractional exponent denotes a power, and the denominator, a root.

For example, $a^{\frac{1}{2}} = \sqrt{a}$; $c^{\frac{2}{3}} = \sqrt[3]{c^2}$; $x^{\frac{m}{n}} = \sqrt[n]{x^m}$; $2c^{\frac{m}{5}} = 2\sqrt[5]{c^m}$, the exponent $\frac{m}{5}$ applying only to the c.

132. The meaning of **negative exponents** may be illustrated as follows: Let it be required to divide a^4 by a^2. By Art. **70**, we have, $\frac{a^4}{a^2} = a^{4-2} = a^2$; likewise, $a^2 \div a^2 = \frac{a^2}{a^2} = a^{2-2} = a^0$; also, $a^0 \div a^2 = \frac{a^0}{a^2} = a^{0-2} = a^{-2}$, etc. From this, it will be seen that $\frac{a^0}{a^2} = a^{-2}$; but, by Art. **71**, $a^0 = 1$, so that 1 may be written for a^0, in this expression; thus, $\frac{a^0}{a^2} = \frac{1}{a^2} = a^{-2}$. Hence,

A quantity affected with a negative exponent denotes the reciprocal of the same quantity affected with an equal positive exponent. (Art. **98.**)

133. Since in $\frac{1}{a^2} = a^{-2}$, the a^{-2} changes to a^2 when placed in the denominator, we may state the following principle:

A factor may be changed from the numerator to the denominator, or from the denominator to the numerator, if the sign of its exponent be changed.

For example, $\frac{n^{-3}}{ab} = \frac{1}{abn^3}$; $\frac{n}{ab^{-\frac{1}{2}}} = \frac{nb^{\frac{1}{2}}}{a}$; $\frac{x^{-\frac{2}{3}}}{5y^{-1}} = \frac{y}{5x^{\frac{2}{3}}}$, etc. In the last, the positive exponent 1 of the y is not written.

EXAMPLE.—Express, with positive exponents,

$$a^{-1}b^{-2}c^{3}+a^{-2}b^{-\frac{3}{4}}c^{-\frac{1}{3}}+a^{3}b^{-2}.$$

SOLUTION.—Since these terms may be taken as fractions, with 1 for the denominators, we have, by transferring the letters with negative exponents to the denominators,

$$a^{-1}b^{-2}c^{3}+a^{-2}b^{-\frac{3}{4}}c^{-\frac{1}{3}}+a^{3}b^{-2}=\frac{c^{3}}{ab^{2}}+\frac{1}{a^{2}b^{\frac{3}{4}}c^{\frac{1}{3}}}+\frac{a^{3}}{b^{2}}.\quad \text{Ans.}$$

134. The student must note very carefully that factors of an entire term only can be changed from numerator to denominator, or vice versa, and that when thus changed they become factors of the whole of the other term. Thus, in $\frac{a}{bc^{-2}+d}$, c^{-2} *cannot* be transferred to the numerator by merely changing the sign of the exponent. The exponent may, however, be made positive by multiplying both terms by c^{2}; thus, $\frac{a\times c^{2}}{(bc^{-2}+d)\times c^{2}}=\frac{ac^{2}}{b+c^{2}d}$. In $\frac{ac^{-2}}{b+d}$, if we transfer the c^{-2}, it becomes $\frac{a}{c^{2}(b+d)}$, c^{2} becoming a factor of the *entire* denominator.

EXAMPLE.—Clear $x^{2}y^{-2}z^{-1}+\frac{2xy}{y^{-1}-x^{3}}-\frac{3a^{-1}b^{-2}c^{3}}{a^{2}+b}$ of negative exponents.

SOLUTION.—Treat each term of the expression separately. $x^{2}y^{-2}z^{-1}=\frac{x^{2}y^{-2}z^{-1}}{1}$; changing the factors with negative exponents to the denominator, and at the same time changing the signs of the exponents, we have $\frac{x^{2}}{y^{2}z}$. In $\frac{2xy}{y^{-1}-x^{3}}$, y^{-1} is not a factor of the whole denominator, so we must multiply both terms of the fraction by the reciprocal of y^{-1} or y; thus, $\frac{2xy\times y}{(y^{-1}-x^{3})\times y}=\frac{2xy^{2}}{1-x^{3}y}$. In $\frac{3a^{-1}b^{-2}c^{3}}{a^{2}+b}$, a^{-1} and b^{-2} are factors of the entire numerator, so we write them as factors of the entire denominator, with the signs of the exponents changed; thus, $\frac{3a^{-1}b^{-2}c^{3}}{a^{2}+b}=\frac{3c^{3}}{ab^{2}(a^{2}+b)}=\frac{3c^{3}}{a^{3}b^{2}+ab^{3}}$. Hence,

$$x^{2}y^{-2}z^{-1}+\frac{2xy}{y^{-1}-x^{3}}-\frac{3a^{-1}b^{-2}c^{3}}{a^{2}+b}=\frac{x^{2}}{y^{2}z}+\frac{2xy^{2}}{1-x^{3}y}-\frac{3c^{3}}{a^{3}b^{2}+ab^{3}}.\quad \text{Ans.}$$

EXAMPLE.—Solve the following:

$$a^3 \times a^{-1};\ n \times n^{-\frac{1}{4}};\ 2c^{-\frac{2}{3}} \times \frac{1}{-3\sqrt[3]{c^2}};\ c^{\frac{n}{m}} \div c^{\frac{2n}{m}};\ x^2 \div \sqrt[5]{x^2}.$$

Write the answers with positive exponents.

SOLUTION.— $a^3 \times a^{-1} = a^{3+(-1)} = a^{3-1} = a^2$. Ans.

$$n \times n^{-\frac{1}{4}} = n^{1+(-\frac{1}{4})} = n^{1-\frac{1}{4}} = n^{\frac{3}{4}}. \text{ Ans.}$$

$$2c^{-\frac{2}{3}} \times \frac{1}{-3\sqrt[3]{c^2}} = \frac{2c^{-\frac{2}{3}}}{1} \times -\frac{1}{3\sqrt[3]{c^2}} = \frac{2}{c^{\frac{2}{3}}} \times -\frac{1}{3c^{\frac{2}{3}}} = -\frac{2 \times 1}{c^{\frac{2}{3}} \times 3c^{\frac{2}{3}}}$$

$$= -\frac{2}{3c^{\frac{2}{3}+\frac{2}{3}}} = -\frac{2}{3c^{\frac{4}{3}}}. \text{ Ans.}$$

$$c^{\frac{n}{m}} \div c^{\frac{2n}{m}} = c^{\frac{n}{m}-\frac{2n}{m}} = c^{\frac{n-2n}{m}} = c^{\frac{-n}{m}} = c^{-\frac{n}{m}} = \frac{1}{c^{\frac{n}{m}}}. \text{ Ans.}$$

$$x^2 \div \sqrt[5]{x^2} = x^2 \div x^{\frac{2}{5}} = x^{2-\frac{2}{5}} = x^{\frac{10}{5}-\frac{2}{5}} = x^{\frac{8}{5}}. \text{ Ans.}$$

135. A letter may be raised to any power by multiplying its exponent by the index of the power. Thus, $(x^2)^3 = x^2 \times x^2 \times x^2 = x^{2\times 3} = x^6$, $(a^2b^3)^5 = a^{2\times 5}b^{3\times 5} = a^{10}b^{15}$, $(a^2)^{\frac{1}{2}} = a$, $(a^2)^{-\frac{1}{2}} = a^{-1}$, etc.

EXAMPLE.—Find the values of the following: $(a^{-1})^{-\frac{1}{2}}$; $(cd^{-2})^{\frac{5}{2}}$; $(x^a)^{-b} \div (x^{-a})^{-b}$.

SOLUTION.—In the first, multiplying the exponents, we have $-1 \times -\frac{1}{2} = \frac{1}{2}$. Hence, $(a^{-1})^{-\frac{1}{2}} = a^{\frac{1}{2}}$, or $\sqrt{a}$. Ans. In like manner, $(cd^{-2})^{\frac{5}{2}} = c^{\frac{5}{2}}d^{-5}$, Ans., since $1 \times \frac{5}{2} = \frac{5}{2}$, and $-2 \times \frac{5}{2} = -5$.

In the next one, $(x^a)^{-b} = x^{-ab}$ and $(x^{-a})^{-b} = x^{ab}$. Dividing,

$$x^{-ab} \div x^{ab} = x^{-ab-ab} = x^{-2ab}. \text{ Ans.}$$

136. A root of a letter affected with an exponent is extracted by dividing the exponent by the index of the root. Thus, $\sqrt[4]{y^{12}} = y^{\frac{12}{4}} = y^{12\div 4} = y^3$. From this principle, the following rule may be deduced.

To extract any root of a monomial:

Rule.—*Extract the required root of the numerical coefficient, and divide the exponent of each letter by the index of the root. Make the sign of every even root of a positive quantity* ±, *and the sign of every odd root of any quantity the same as that of the quantity.*

EXAMPLE.—Find the value of $\sqrt[4]{256a^4b^{12}c^8}$.

SOLUTION.—The 4th root of 256 is 4. The exponent of a in the root is $4 \div 4 = 1$; of b, $12 \div 4 = 3$; and of c, $8 \div 4 = 2$. As this is an even root of a positive quantity, the sign should be $\pm$. Hence,

$$\sqrt[4]{256a^4b^{12}c^8} = \pm 4ab^3c^2. \quad \text{Ans.}$$

EXAMPLE.—Find the value of $\sqrt[3]{\dfrac{27m^3x^9}{a^9b^6c^{12}}}$.

SOLUTION.— $\sqrt[3]{27m^3x^9} = 3mx^3$; $\sqrt[3]{a^9b^6c^{12}} = a^3b^2c^4$. The quantity is positive, and, as this is an odd root, its sign must be the same, or positive.

Hence, $\sqrt[3]{\dfrac{27m^3x^9}{a^9b^6c^{12}}} = \dfrac{3mx^3}{a^3b^2c^4}$. Ans.

EXAMPLES FOR PRACTICE.

137. Clear the following of negative exponents:

1. $x^2y^{-2}z^{-\frac{1}{2}}$. Ans. $\dfrac{x^2}{y^2z^{\frac{1}{2}}}$.

2. $3a^{-1}b + \dfrac{2a}{b^{-3}c^{-1}} + c^{-1}$. Ans. $\dfrac{3b}{a} + 2ab^3c + \dfrac{1}{c}$.

3. $\dfrac{4a^{-2}(c+d)}{2c+d}$. Ans. $\dfrac{4(c+d)}{a^2(2c+d)}$.

Express the following without radical signs:

4. $\sqrt[3]{b^{-2}}$. Ans. $(b^{-2})^{\frac{1}{3}}$ or $b^{-\frac{2}{3}}$.

5. $4a\sqrt{a^{-1}b^3}$. Ans. $4aa^{-\frac{1}{2}}b^{-\frac{3}{2}} = 4a^{-\frac{1}{2}}b^{-\frac{3}{2}}$.

Find the values of the following:

6. $m^{\frac{1}{2}} \times m^{\frac{1}{6}}$. Ans. $m^{\frac{1}{3}}$.

7. $2ab^{\frac{1}{2}} \times a^{\frac{1}{2}}b$. Ans. $2a^{\frac{1}{2}}b^{\frac{3}{2}}$.

8. $c^{\frac{n}{2}} \div \sqrt{c^{-n}}$. Ans. c^n.

9. $2x^{-2} \div (x^2)^{-\frac{1}{2}}$ Ans. $2x^{-1}$.

10. $\left(cd^{-\frac{n}{m}}\right)^{2m} \times \sqrt{d^{4n}}$. Ans. c^{2m}.

EQUATIONS.

DEFINITIONS.

138. As defined in Art. **5**, an **equation** is a statement of equality between two expressions, as $x + 6 = 14$.

Every equation has two parts, called the **first** and the

second member. The first member is the part on the left of the sign of equality, and the second member the part on the right of that sign. In $x+6 = 14$, $x+6$ is the first member, and 14 is the second member.

139. Equations usually consist of **known** and **unknown quantities**; that is, of quantities whose values are given, and of quantities whose values are not given, but are to be found. Thus, in $x+6 = 14$, 6 and 14 are known quantities, and x is unknown; but since by the statement of the equation, $x+6$ must equal 14, x must have such a value that when added to 6 the sum will be 14. Hence, the value of x is fixed for this particular case, and in a similar manner the value of a single unknown quantity in any equation is fixed by the relations that it bears to the known quantities, and this value can usually be found.

140. **To solve an equation** is to find the value of the unknown quantity. This is done by a series of **transformations** by which the first member becomes the unknown quantity, and the second member becomes a known quantity, which is, therefore, the value of the unknown quantity.

TRANSFORMATIONS.

141. In transforming an equation, the equality of its members must be preserved; otherwise the existing relations between the known and unknown quantities will be destroyed. Transformations are based upon the following principles:

142. In any equation:

I. The same quantity may be added to both members. For example, if 2 be added to both members of $x^2 = 16$, the members of the resulting equation, $x^2+2 = 18$, will be equal.

II. The same quantity may be subtracted from both members. Thus, if $x^2 = 16$, then $x^2-2 = 14$.

III. Both members may be multiplied or both divided by the same quantity. Thus, if $x^2 = 16$, then $2x^2 = 32$ and $\frac{x^2}{2} = 8$.

IV. Both members may be raised to the same power. Thus, if $x^2 = 16$, then $x^4 = 256$.

V. Like roots of both members may be extracted. Thus, if $x^2 = 16$, then $x = 4$.

A little thought will show that none of these operations will destroy the quality of the members. In the equation $16 = 16$, for example, by I, $16+2 = 16+2$; by II, $16-2 = 16-2$; by III, $16\times 2 = 16\times 2$, etc. It is to be observed, however, that after any transformation, the *members* do not equal their original values.

143. To **transpose** a term in an equation is to change it from one member of an equation to the other. *A term may be transposed to the other member of an equation, if its sign be changed.* Thus, in the equation $2x+5 = 13$, let it be required to transpose $+5$ to the second member; changing its sign, we have $2x = 13-5$, or $2x = 8$. For, subtract 5 from both members and we have $2x+5-5 = 13-5$, or $2x = 8$. Suppose we had $2x-5 = 13$; changing the sign of -5, and placing it in the second member, we have $2x = 18$; for $2x-5+5 = 13+5$, or $2x = 18$.

144. *When the same term appears in both members of an equation, with the same sign, it may be dropped from each.* This is called **cancelation.** Thus, in $x+a = 16+a$ we may cancel a, and have $x = 16$, for subtracting a from both members $x+a-a = 16+a-a$; hence, $x = 16$. In $x-a = 16-a$, $x-a+a = 16-a+a$, and $x = 16$. But, in $x-a = 16+a$, the a's will not cancel, since they have different signs.

145. Changing Signs.—It is sometimes desirable to change the sign of a term in one of the members of an equation. This may be effected by multiplying both members

by -1. This gives the same result as changing the signs of all the terms of both members; thus, $x+a+3 = a-x-7$ may be changed to $-x-a-3 = -a+x+7$. According to Art. **142,** III, this transformation does not destroy the equality of the members.

146. Clearing of fractions is usually necessary before performing any operations.

Rule.—*To clear an equation of fractions, multiply each term of the equation by the common denominator of all the fractions.*

EXAMPLE.—Clear of fractions $x+\frac{x}{2}+\frac{3x}{4}-\frac{2x}{6} = 10$.

SOLUTION.—The common denominator of all the fractions is 12; multiplying each term by 12, we have

$$12x+6x+9x-4x = 120.$$

EXAMPLE.—Clear of fractions $\frac{2x}{x+2} = \frac{1}{2}-\frac{3x+2}{x^2-4}$.

SOLUTION.—The common denominator is $2(x^2-4)$; multiplying through by this we get $4x(x-2) = (x^2-4)-2(3x+2)$; removing the parentheses this becomes $4x^2-8x = x^2-4-6x-4$.

Where a fraction is preceded by a minus sign, care must be taken to change the sign of every term of the numerator when clearing of fractions. (See Art. **110.**)

EXAMPLES FOR PRACTICE.

147. Clear the following equations of fractions:

1. $x+\frac{3x}{4}+\frac{5}{7} = 16-\frac{2}{x}$. Ans. $28x^2+21x^2+20x = 448x-56$.

2. $\frac{x}{4}-\frac{x-3}{2} = \frac{a}{6}$. Ans. $3x-6x+18 = 2a$.

3. $\frac{x}{a-b}-x = \frac{a-b}{a+b}-1$.
Ans. $ax+bx-a^2x+b^2x = a^2-2ab+b^2-a^2+b^2$.

4. $\frac{1}{a-b} = \frac{x}{a-b}-\frac{a+b}{x}$. Ans. $x = x^2-a^2+b^2$.

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

148. Simple equations, when reduced to their simplest form, contain the first power only of the unknown quantity. When there is but one unknown quantity, it is usually represented by x. Numbers and the *first* letters of the alphabet are used for known quantities.

149. To solve simple equations with one unknown quantity:

Rule.—*Clear of fractions. Remove all signs of aggregation. Transpose all terms containing the unknown quantity to the first member; all others to the second member. Combine the first member into one term, and simplify the second member. Divide both members by the coefficient of the unknown quantity.*

In some cases, this order of operations may be changed to advantage.

150. To verify the result, substitute the value of the unknown quantity in the original equation, which should then reduce so that both members will be alike. When this occurs the equation is said to be **satisfied.**

EXAMPLE.—Solve $2x+5 = 25-3x$.

SOLUTION.—Transposing the unknown quantities to the first member, we have

$$2x+3x+5 = 25.$$

Transposing the known quantities to the second member, we have

$$2x+3x = 25-5.$$

Combining like terms, $5x = 20.$

Dividing by 5, $x = 4.$ Ans.

Now, if the substituting of 4 for x will satisfy the original equation, we know our answer is correct. Thus, substituting 4 for x, we have

$$2\times 4+5 = 25-3\times 4.$$
$$8+5 = 25-12.$$
$$13 = 13.$$

Hence the equation is satisfied, and our result is correct.

EXAMPLE.—Solve $16-x-\{7x-[8x-(9x-\overline{3x-6x})]\} = 0$.

SOLUTION.—Removing the symbols of aggregation (Art. **52**),

$$16-x-7x+8x-9x+3x-6x = 0.$$

Transposing 16 to the second member,

$$-x-7x+8x-9x+3x-6x = -16.$$

Combining like terms, $-12x = -16.$

Dividing both members by -12,

$$x = \tfrac{16}{12} = 1\tfrac{1}{3}. \quad \text{Ans.}$$

EXAMPLE.—Solve $\dfrac{2x+2}{2}+\dfrac{1}{4}=\dfrac{8-6x}{5}+\dfrac{2(6x+7)}{8}$.

SOLUTION.—Simplifying the first and the last term of the equation, it becomes

$$x+1+\frac{1}{4}=\frac{8-6x}{5}+\frac{6x+7}{4}.$$

Clearing of fractions, we have

$$20x+20+5 = 32-24x+30x+35.$$

Transposing, $20x+24x-30x = 32+35-20-5.$

Combining, $14x = 42.$

Dividing by 14, $x = 3.$

EXAMPLE.—Solve $\dfrac{3}{1-x}-\dfrac{2}{1+x}+\dfrac{1}{1-x^2} = 0.$

SOLUTION.—Clearing of fractions by multiplying by $1-x^2$,

$$3(1+x)-2(1-x)+1 = 0.$$

$$3+3x-2+2x+1 = 0.$$

Uniting terms, $5x = -2.$

$$x = -.4. \quad \text{Ans.}$$

NOTE.— 0 multiplied or divided by any number = 0.

151. Literal equations are those in which the known quantities are represented partially or wholly by letters. In solving such equations, we cannot always combine the unknown quantities into one term.

EXAMPLE.—Solve $2ax-3b = x+c-3ax.$

SOLUTION.—Transposing the terms containing the unknown quantities to the first member and the remaining terms to the second member, and combining like terms,

$$5ax-x = 3b+c.$$

Factoring $5ax-x$ to bring x alone in the first member,

$$(5a-1)x = 3b+c.$$

The coefficient of x is now $5a-1$, this being considered as one quantity.

Dividing by $5a-1$, $x = \dfrac{3b+c}{5a-1}.$ Ans.

PROOF.—Since the original equation is equivalent to $5ax - x = 3b + c$, it will be sufficient to satisfy this equation. Hence, substituting the value of x,

$$\frac{5a(3b+c)}{5a-1} - \frac{3b+c}{5a-1} = 3b+c, \text{ or } \frac{(5a-1)(3b+c)}{5a-1} = 3b+c.$$

Canceling the $5a-1$, $3b+c = 3b+c$.

EXAMPLE.—Solve $\dfrac{3x+1}{x+1} = \dfrac{3bx-2a+c}{b(x+1)-a}$.

SOLUTION.—Clearing of fractions,

$$(3x+1)[b(x+1)-a] = (x+1)(3bx-2a+c), \text{ or}$$
$$3bx(x+1) - 3ax + b(x+1) - a = 3bx(x+1) - (2a-c)(x+1).$$

Canceling $3bx(x+1)$ from both members,

$$-3ax + bx + b - a = -2ax + cx - 2a + c.$$

Transposing and uniting terms,

$$-ax + bx - cx = -a - b + c.$$

Changing signs and factoring,

$$(a-b+c)x = a+b-c.$$

$$x = \frac{a+b-c}{a-b+c}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

152. Solve the following:

1. $16 - 3x = 13 - 6x$. Ans. $x = -1$.
2. $3(4x-5)+6 = 1+2x$. Ans. $x = 1$.
3. $6(5-2x) = 6-2(x-2)$. Ans. $x = 2$.
4. $\dfrac{2x}{3} - \dfrac{4x}{3} = 5 - \dfrac{3x}{4}$. Ans. $x = 60$.
5. $\dfrac{x+1}{3} - \dfrac{x+4}{5} = 16 - \dfrac{x+3}{4}$. Ans. $x = 41$.
6. $\dfrac{x}{3} - \dfrac{x^2-5x}{3x-7} = \dfrac{2}{3}$. Ans. $x = -7$.
7. $\dfrac{5-2x}{x+1} - \dfrac{3-2x}{x+4} = 0$. Ans. $x = 4\frac{1}{4}$.
8. $2x - 4a = 3ax + a^2 - a^2x$. Ans. $x = \dfrac{a^2+4a}{a^2-3a+2}$.
9. $\dfrac{ax+2x}{5a} - \dfrac{a^2+4a+4}{4b} = 0$. Ans. $x = \dfrac{5a^2+10a}{4b}$.

SUGGESTION.—Transposing the second term to the second member,

$$\frac{ax+2x}{5a} = \frac{a^2+4a+4}{4b} = \frac{(a+2)^2}{4b}.$$

Multiplying both sides by $5a$,

$$ax + 2x = \frac{5a(a+2)^2}{4b}.$$

Solving for x,

$$x = \frac{5a(a+2)^2}{(a+2)4b} = \frac{5a(a+2)}{4b} = \frac{5a^2+10a}{4b}.$$

10. $\frac{a(c^2+x^2)}{cx} = ab + \frac{ax}{c}$. Ans. $x = \frac{c}{b}$.

PROBLEMS LEADING TO SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

153. There are two steps in the solution of problems by algebra:

I.—The relations which exist between the known and the unknown quantities, that is, between those whose values are given in the problem and those whose values are required, must be stated by one or more equations. This is called the **statement** of the problem.

II.—The resulting equation or equations must be solved, giving the values of the required quantities.

It will thus be seen that by the algebraic method, the answer to a problem is used in the solution and operated upon as though it were a known quantity, which is one great advantage over the arithmetical method.

154. The ability to state a problem by means of an equation depends upon the ingenuity of the operator and his ability to reason, rather than upon his knowledge of algebra. No definite rule can be given for making the statement, but in general, where there is only one unknown quantity in a problem:

Decide what quantity it is whose value is to be found. This will be the unknown quantity, or the answer. Then represent the unknown quantity by x and form an equation that will indicate the relations between the known and the unknown quantities as stated in the problem.

The equation will also indicate the operations that would be performed in proving the statement made in the problem

were the answer known. Hence, the equation may often be formed by noticing what operations would be performed upon the answer in proving.

155. The following examples are illustrations of the statement and the solution of algebraic problems; they should be studied carefully.

EXAMPLE.—Find such a number that, when 14 is added to its double, the sum shall be 30.

SOLUTION.—The quantity whose value is required is the number itself. As this is the unknown quantity, let x = the number, whence $2x$ must be double the number. Now the problem states that when 14 is added to double the number the sum will be 30. In other words, when 14 is added to $2x$, the sum will be 30. Hence, the statement of the problem in the form of an equation is

$$2x + 14 = 30;$$

whence, solving, $$x = 8. \quad \text{Ans.}$$

EXAMPLE.—Find a number which, when multiplied by 4, will exceed 40 as much as it is now below 40.

SOLUTION.—Let x = the required number, which, when multiplied by 4, becomes $4x$. According to the conditions of the problem, the amount by which 4 times the required number, or $4x$, exceeds 40 is equal to the amount that the number itself, or x, is below 40.

But $4x - 40$ is the amount by which $4x$ exceeds 40, and $40 - x$ is the amount by which x is below 40.

Hence, by the conditions, we have the statement,

$$4x - 40 = 40 - x.$$

Transposing and uniting, $5x = 80,$

or $$x = 16. \quad \text{Ans.}$$

EXAMPLE.—Two loads of brick together weigh 4,000 lb.; but if 500 lb. be transferred from the smaller to the larger load, the latter will weigh 7 times as much as the former. How much does each load weigh?

SOLUTION.—If the weights of the two loads were known and it was desired to prove the correctness of the example, we should add 500 lb. to the weight of the larger load and subtract 500 lb. from the weight of the smaller load, as stated in the example. The larger load should then weigh 7 times as much as the smaller. To obtain the equation the same thing is done by letting x = the weight of one load, whence $4,000 - x$ is equal to the weight of the other load.

Let x = the weight of the smaller load.

Then, $4,000 - x$ = the weight of the larger load.

Also, $x - 500$ = the weight of the smaller load after transferring 500 lb.

And $4,000 - x + 500$ = the weight of the larger load after transferring 500 lb.

By the conditions, the larger load now weighs 7 times as much as the smaller.

Hence, $7(x - 500) = 4,000 - x + 500.$

Solving, $7x - 3,500 = 4,500 - x,$

or $8x = 8,000;$

whence, $x = 1,000$ lb. = weight of smaller load,

and $4,000 - x = 3,000$ lb. = weight of larger load. } Ans.

PROOF.— $1,000 - 500 = 500$ = weight of the smaller load, and $3,000 + 500 = 3,500$ = weight of the larger load after the 500 pounds have been transferred; $3,500 \div 500 = 7$.

Until the student has obtained considerable proficiency in solving problems of this kind, it is a good plan to prove all problems.

EXAMPLE.—The circumference of the fore wheel of a carriage is 10 feet, and of the rear wheel, 12 feet. What distance has the carriage traveled, when the fore wheel has made 8 more turns than the rear wheel?

SOLUTION.—In this example the distance traveled is not known, but is required to be found. Suppose that the distance is known, and that it is equal to x feet, and that we wish to see whether the statement is true that the fore wheel makes 8 more revolutions than the rear wheel in passing over x feet. The number of revolutions of the fore wheel is evidently $\frac{x}{10}$, and of the rear wheel, $\frac{x}{12}$. The example states that the difference between them is equal to 8.

Hence, $\frac{x}{10} - \frac{x}{12} = 8.$ (1)

Solving for x, $12x - 10x = 960,$

or $2x = 960,$

and $x = 480$ feet. Ans.

PROOF.— $\frac{480}{10} = 48$ = revolutions of fore wheel.

$\frac{480}{12} = 40$ = revolutions of rear wheel.

$48 - 40 = 8.$ Compare this proof with (1).

EXAMPLE.—A water cistern connected with three pipes can be filled by one of them in 80 minutes, by another in 200 minutes, and by the third in 300 minutes. In what time will the cistern be filled when all three pipes are open at once?

SOLUTION.—Here the unknown quantity is the number of minutes required to fill the cistern by all three pipes together. Supposing this to be x minutes, the example may be proved by noticing that the sum of the quantities of water flowing through each pipe separately in a given length of time, as 1 minute, must be equal to the quantity flowing through all three together in the same length of time. According to the problem, the quantity discharged by the first pipe in 1 minute would be $\frac{1}{80}$, by the second $\frac{1}{200}$, and by the third $\frac{1}{300}$ of the contents of the cistern. In like manner the quantity discharged by all three at once in 1 minute would be $\frac{1}{x}$. Then, if the example is stated correctly, we must have

$$\frac{1}{80}+\frac{1}{200}+\frac{1}{300}=\frac{1}{x}.$$

Clearing of fractions, $x(30+12+8) = 2{,}400,$

or $50x = 2{,}400;$

whence, $x = 48$ minutes. Ans.

EXAMPLE.—A man rows a boat a certain distance *with* the tide, at the rate of $6\frac{2}{3}$ miles an hour, and returns at the rate of $3\frac{1}{3}$ miles an hour, *against* a tide half as strong. If the man is pulling at a uniform rate, what is the velocity of the stronger tide?

SOLUTION.—If the following statement is not clear, the student should reason it out for himself in a manner similar to that used in the last three examples.

Let x = number of miles per hour that the stronger tide is running; then $\frac{x}{2}$ = number of miles per hour that the weaker tide is running.

Hence, $6\frac{2}{3}-x$ and $3\frac{1}{3}+\frac{x}{2}$ are expressions for the rate at which the man is pulling. But, as he is pulling at a constant rate all the time, these expressions must be equal. Hence,

$$6\tfrac{2}{3}-x = 3\tfrac{1}{3}+\frac{x}{2},$$

or $$\frac{20}{3}-x = \frac{10}{3}+\frac{x}{2}.$$

Clearing of fractions, $40-6x = 20+3x,$

or $-9x = -20;$

whence, $x = 2\frac{2}{9}$ miles per hour. Ans.

EXAMPLES FOR PRACTICE.

156. Solve the following examples:

1. The greater of two numbers is four times the lesser number, and their sum is 400; what are the numbers? Ans. 80 and 320.

2. A farmer has 108 animals, consisting of horses, sheep, and cows. He has four times as many cows as horses, lacking 8, and five times as many sheep as horses, lacking 4; how many has he of each kind?

Ans. { 12 horses. 40 cows. 56 sheep.

3. A can do a piece of work in 8 days, and B can do it in 10 days; in what time can they do it working together? Ans. $4\frac{4}{9}$ days.

4. Find five consecutive numbers whose sum is 150.

Ans. 28, 29, 30, 31, and 32.

5. A boat whose rate of sailing is 6 miles per hour in still water moves down a stream which flows at the rate of 3 miles per hour, and returns, making the round trip in 8 hours; how far did it go down the stream? Ans. 18 miles.

THE TRIGONOMETRIC FUNCTIONS.

DEFINITIONS.

157. Plane trigonometry treats of the solution of plane triangles.

Every triangle has six parts—three angles and three sides. When three of these parts are given, if one part at least is a side, the other three can be found. This process of finding the unknown parts of a triangle is the **solution** of the triangle.

158. The **complement** of an angle is the difference between 90° and the angle. Thus, the complement of an angle of 35° is an angle of 55°, because $90° - 35° = 55°$. In a right-angled triangle, the right angle is 90°; since the sum of the three angles of the triangle is 180°, the sum of the two acute angles is $180° - 90° = 90°$. Therefore, each acute angle of a right-angled triangle is the complement of the other acute angle.

159. The **supplement** of an angle is the difference between 180° and the angle. Thus, the supplement of an angle of 35° is an angle of 145°, because $180° - 35° = 145°$.

160. The solution of a triangle is accomplished by means of the **trigonometric functions.** These functions are the ratios of the sides of a right-angled triangle; the most important of these functions are the *sine*, *cosine*, *tangent*, and *cotangent*. These are abbreviated to *sin*, *cos*, *tan*, and *cot*.

161. In the right-angled triangle $A\,B\,C$, Fig. 2, the sides a, b, and c are opposite, respectively, to the angles A, B, and C. The hypotenuse is c, and C is the right angle. The short side b is adjacent to angle A and opposite angle B; the short side a is opposite to angle A and adjacent to angle B.

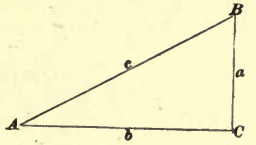

FIG. 2.

Then the trigonometric functions are defined as follows:

162. The **sine** of an angle is the quotient of the opposite side divided by the hypotenuse. Thus, in Fig. 2, $\sin A = \frac{a}{c}$.

163. The **cosine** of an angle is the quotient of the adjacent side divided by the hypotenuse. Thus, $\cos A = \frac{b}{c}$.

164. The **tangent** of an angle is the quotient of the opposite side divided by the adjacent side. Thus, $\tan A = \frac{a}{b}$.

165. The **cotangent** of an angle is the quotient of the adjacent side divided by the opposite side. Thus, $\cot A = \frac{b}{a}$.

166. From the definitions of the functions, we have: $\sin B = \frac{b}{c}$; $\cos B = \frac{a}{c}$; $\tan B = \frac{b}{a}$; and $\cot B = \frac{a}{b}$. Comparing the functions of angle B with those of angle A,

$\sin B = \cos A$ (since each is equal to $\frac{b}{c}$), $\cos B = \sin A$, $\tan B = \cot A$, and $\cot B = \tan A$. It has been shown (Art. **158**) that angle A is the complement of angle B. Therefore, the sine of an angle is equal to the cosine of its complement, and the tangent of an angle is equal to the cotangent of its complement. For example, $\sin 36° = \cos (90° - 36°) = \cos 54°$; $\tan 28° = \cot 62°$; etc.

TRIGONOMETRIC TABLES.

167. Every angular function has a different value for each of the angles between 0° and 90°. The numerical values of these functions are called natural sines, cosines, etc., and are given in the tables of *Natural Sines, Cosines, Tangents, and Cotangents.* In many tables, both natural and logarithmic functions are given. The student should not attempt to use the latter until he thoroughly understands logarithms. The table of natural functions, and its use, will now be explained.

168. Given an Angle, to Find Its Functions.

EXAMPLE.—Let it be required to find the sine, cosine, and tangent of an angle of 37° 24′.

SOLUTION.—Look in the table of Natural Sines along the tops of the pages and find 37°. The left-hand column is marked (′), meaning that the minutes are to be sought in that column, and begin with 0, 1, 2, 3, etc., up to 60. Glancing *down* this column until 24 is found, find opposite this 24 in the column marked *sine*, and headed 37°, the number .60738; then .60738 = sin 37° 24′. In exactly the same manner, find in the column marked *cosine* and headed 37°, the number .79441, which corresponds to cos 37° 24′; or cos 37° 24′ = .79441. So, also, find in the column marked *tangent* and headed 37°, and opposite 24′, the number .76456; hence, tan 37° 24′ = .76456.

169. In most of the tables published, the angles run only from 0° to 45° at the top of the page; to find an angle greater than 45°, *look at the bottom of the page and glance upwards*, **using the extreme right-hand column to**

find minutes, which begin with 0 at the bottom and run upwards, 1, 2, 3, etc. to 60.

EXAMPLE.—Find the sine, cosine, and tangent of 77° 43′.

SOLUTION.—Since this angle is greater than 45°, look in the tables along the bottom of the page, until the column marked 77° is found. Glancing up the column of minutes on the *right*, until 43′ is found, find opposite 43′ in the column marked *sine* (and 77°) at the bottom, the number .97711; this is the sine of 77° 43′, or sin 77° 43′ = .97711. Similarly, in the column marked *cosine*, find opposite 43′ in the right-hand column, the number .21275; this is the cosine of 77° 43′, or cos 77° 43′ = .21275. So, also, find that 4.59283 is the tangent of 77° 43′, or tan 77° 43′ = 4.59283.

170. Let it be required to find the sine of 14° 22′ 26″.

EXPLANATION. — The sine of 14° 22′ 26″ lies between sin 14° 22′ and sin 14° 23′. Sin 14° 22′ = .24813; sin 14° 23′ = .24841 ; difference = .00028. Neglect for the moment the fact that the functions are decimal fractions; then the difference between sin 14° 22′ and sin 14° 23′, that is, the difference between 24841 and 24813, is 28, or, corresponding to a difference of 1′ in the angle, there is a difference of 28 in the sine. Now, since 1′ = 60″, the difference between sin 14° 22′ and sin 14° 22′ 26″, that is, the difference corresponding to a difference of 26″ in the angle, must be $\frac{26}{60} \times 28 = 12.1$. Since .1 is less than .5, omit it, and we have 12 as the difference to be added to 24813. 24813 + 12 = 24825 ; taking account now of the fact that the function is a decimal fraction, sin 14° 22′ 26″ = .24825.

In all work with the tables it will always be found most convenient to neglect for the moment the decimal point, consider the difference a whole number, and afterwards locate the decimal point in its proper position.

171. Reference to the table of functions shows that, as the angles increase in magnitude, the sines and tangents increase, while the *cosines and cotangents decrease.* In the above example, therefore, had it been required to find the *cosine* of 14° 22′ 26″, the correction for the 26″ would have been *subtracted* from the cosine of 14° 22′, instead of being added to it.

EXAMPLE.—Find the sine, cosine, and cotangent of 56° 43′ 17″.

SOLUTION.—Sin 56° 43′ = .83597. Sin 56° 44′ = .83613. Since 56° 43′ 17″ is greater than 56° 43′ and less than 56° 44′, the value of the sine of the angle lies between .83597 and .83613; neglecting the decimal character of the functions, the difference = 83613 − 83597 = 16; multiplying this by the fraction $\frac{17}{60}$, $16 \times \frac{17}{60} = 4.53$. Since .53 exceeds .5, we take 5 as the difference to be *added* to 83597. Adding, 83597 + 5 = 83602; hence, sin 56° 43′ 17″ = .83602. Ans.

Cos 56° 43′ = .54878; cos 56° 44′ = .54854; expressed as a whole number, the difference = 54878 − 54854 = 24, and $24 \times \frac{17}{60} = 7$, nearly. Now, since the *cosine* is desired, we must *subtract* this correction from 54878; subtracting, 54878 − 7 = 54871. Hence, cos 56° 43′ 17″ = .54871. Ans.

Cot 56° 43′ = .65646; cot 56° 44′ = .65604; difference = 65646 − 65604 = 42, and $42 \times \frac{17}{60} = 12$, nearly. Now, since the cotangent decreases as the angle increases, we must *subtract* this correction from 65646; thus, 65646 − 12 = 65634. Hence, cot 56° 43′ 17″ = .65634. Ans.

172. Given the Function, to Find the Corresponding Angle.—This is the reverse of the process for finding the function of the angle. If the angle corresponding to the given function is an exact number of degrees and minutes, the function will be found in the table. In such a case, we have simply to find the given function, and take the degrees from the end of the column, and the minutes at the end of the row. If the name of the function is at the top of the column, the number of degrees will be found there also, and the minutes at the left. If the name is at the bottom, the degrees will be at the bottom, and the minutes at the right.

CASE I.

173. *The function is found exactly in the table.*

EXAMPLE.—Find the angle whose sine is .24982.

SOLUTION.—In the table of Sines, we find .24982 in the column under 14°. Since the name of the function is at the top of the column, we take the number of degrees from the top, and the minutes from the left; thus, 14° at the top, and 28′ at the left. Hence, the angle whose sine is .24982 is 14° 28′. Ans.

EXAMPLE.—Find the angle whose cotangent is .68557.

SOLUTION.—In the table of Tangents, we find .68557 under 34°. The name of the function is, however, at the bottom of the page, and we must take the degrees 55°, from that end of the column. The minutes, 34′, are found in the right-hand column. Hence, the angle whose cotangent is .68557 is 55° 34′. Ans.

CASE II.

174. *The function is not found exactly in the table.*

Let it be required to find the angle whose sine is .42531.

EXPLANATION.—Referring to the table of Sines, this number is found to lie between .42525, the sine of 25° 10′, and .42552, the sine of 25° 11′. Neglecting decimals, the difference between these two sines $= 42552 - 42525 = 27$; the difference between 42525, corresponding to the sine of 25° 10′, and 42531, corresponding to the sine of the given angle $= 42531 - 42525 = 6$. Since 27 is the difference for 1′, a difference of 6 corresponds to $\frac{6}{27}$ of 1′; hence, the angle whose sine = .42531 = 25° $10\frac{6}{27}$′.

Since 1′ = 60″, $\frac{6}{27}$ of a minute $= \frac{6}{27} \times 60 = 13.3''$. Therefore, the angle whose sine is .42531 = 25° 10′ 13.3″.

The given function should always be compared with the function of the *angle next lower*, and the correction in seconds should be added to that angle. In the case of the sine and tangent, we take the difference between the given function and the next smaller function appearing in the tables; but with the cosine and cotangent, we take the difference between the given function and the next larger function which appears in the tables.

EXAMPLE.—Find the angle whose cosine is .27052.

SOLUTION.—Looking in the table of Cosines, the given function is found to belong to an angle greater than 45°, and, hence, must be sought for in the columns marked *cosine* at the bottom of the page. It is found between the numbers .27060 = cos 74° 18′ and .27032 = cos 74° 19′. The difference between the two is .27060 − .27032 = .00028, or 28, neglecting decimals. The cosine of the *smaller angle*, or 74° 18′, is .27060, and the difference between this and the given cosine is .27060 − .27052 = .00008, or 8, neglecting decimals.

Hence, $\frac{8}{28} \times 60 = 17.14''$, nearly, and the angle whose cosine is .27052 = 74° 18′ 17.14″, or cos 74° 18′ 17.14″ = .27052. Ans.

EXAMPLE.—Find the angle whose tangent is 2.15841.

SOLUTION.— 2.15841 falls between 2.15760 = tan 65° 8′, and 2.15925 = tan 65° 9′. The difference = 2.15925 − 2.15760 = .00165, or 165, considered as a whole number. 2.15841 − 2.15760 = 81, neglecting the decimal. $\frac{81}{165} \times 60 = 29.5''$, nearly, and the angle whose tangent is 2.15841 = 65° 8′ 29.5″, or tan 65° 8′ 29.5″ = 2.15841.

EXAMPLES FOR PRACTICE.

175. Solve the following examples:

1. Find the (*a*) sine, (*b*) cosine, and (*c*) tangent of 48° 17′.

Ans. (*a*) .74644. (*b*) .66545. (*c*) 1.12172.

2. Find the (*a*) sine, (*b*) cosine, and (*c*) cotangent of 13° 11′ 6″.

Ans. (*a*) .22810. (*b*) .97364. (*c*) 4.26855.

3. Find the (*a*) sine, (*b*) cosine, and (*c*) tangent of 72° 0′ 1.8″.

Ans. (*a*) .95106. (*b*) .30901. (*c*) 3.07777.

4. (*a*) Of what angle is .26489 the sine, (*b*) of what is it the cosine, and (*c*) of what is it the cotangent?

Ans. (*a*) 15° 21′ 37.2″. (*b*) 74° 38′ 22.8″. (*c*) 75° 9′ 49″.

5. (*a*) Of what angle is .68800 the sine, (*b*) of what the cosine, and (*c*) of what the tangent?

Ans. (*a*) 43° 28′ 20″. (*b*) 46° 31′ 40″. (*c*) 34° 31′ 40.5″.

SOLUTION OF TRIANGLES.

RIGHT-ANGLED TRIANGLES.

176. When any side and one of the acute angles of a right-angled triangle (also called *right triangle*) are given, the remaining sides and angles may be found; also, when any two sides are given, the remaining parts may be found.

177. The following relations between the sides and angles of a right triangle are derived directly from the definitions of the trigonometric functions:

I. *Side opposite an angle* = *hypotenuse* × *sine of angle.*

II. *Side adjacent* = *hypotenuse* × *cosine.*

III. *Side opposite* = *side adjacent* × *tangent.*

IV. *Side adjacent* = *side opposite* × *cotangent.*

V. $\text{Hypotenuse} = \dfrac{\text{side opposite}}{\text{sine}}$.

VI. $\text{Hypotenuse} = \dfrac{\text{side adjacent}}{\text{cosine}}$.

These relations are sufficient to find the sides. The angles may be found from the sides by the relations given in Arts. **162** to **165.** To show the application of these relations to the solution of triangles, a number of examples are given.

178. Case I.—*The hypotenuse and an acute angle being given, to find the remaining parts:*

EXAMPLE.—In Fig. 3, the hypotenuse AB of the right-angled triangle ACB is 24 feet and the angle A is 29° 31′; to find the sides AC and BC and the angle B.

NOTE.—When working examples of this kind, construct the figure, and mark the known parts. This is a great help in solving the example. Hence, in the figure draw the angle A to represent an angle of 29° 31′, and complete the right-angled triangle ACB, right-angled at C, as shown. Mark the angle A and the hypotenuse, as is done in the figure.

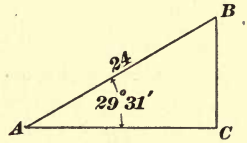

FIG. 3.

SOLUTION.—Angle $B = 90° - 29° 31′ = 60° 29′$. AC is the side adjacent to angle A. Hence, from relation II, Art. **177**, AC, or side adjacent = hypotenuse × cosine = $24 \times \cos 29° 31′$. In the table of Natural Cosines we find the cosine of 29° 31′ to be .87021. Therefore, $AC = 24 \times .87021 = 20.89$ feet, nearly.

To find the side BC, we use relation I, Art. **177**. BC, or side opposite = hypotenuse × sine = $24 \times \sin 29° 31′$. The sine of 29° 31′ is .49268; hence, $BC = 24 \times .49268 = 11.82$ feet, nearly.

Ans. { Angle $B = 60° 29′$.
Side $AC = 20.89$ ft.
Side $BC = 11.82$ ft.

179. Case II.—*An acute angle and one of the short sides being given, to determine the remaining parts:*

EXAMPLE.—In Fig. 3, suppose the side AC to be 75 feet and the angle A to be 32° 24′. The sides BC, AB, and the angle B are required.

SOLUTION.—Angle $B = 90° - 32° 24′ = 57° 36′$. To find BC, we have relation III, Art. **177**, BC, side opposite = side adjacent × tangent = 75 × tan 32° 24′. Referring to the table of Natural Tangents, the tangent of 32° 24′ = .63462. $BC = 75 \times .63462 = 47.6$ ft., nearly. To find the hypotenuse AB, we use relation VI, Art. **177**, hypotenuse $= \frac{\text{side adjacent}}{\text{cosine}}$. $AB = \frac{AC}{\cos A} = \frac{75}{\cos 32° 24′} = \frac{75}{.84433} = 88.81$ ft.

Ans. { Angle $B = 57° 36′$. Side $BC = 47.6$ ft. Side $AB = 88.81$ ft.

180. Case III.—*Two sides being given, to find the third side and the acute angles:*

EXAMPLE.—In the right-angled triangle ABC, Fig. 4, right-angled at C, $AC = 18$ and $BC = 15$; to find AB and the angles A and B.

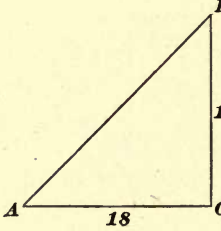

FIG. 4.

SOLUTION.—According to the definition of the tangent, Art. **164**,

tangent $A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{15}{18} = .83333$.

Looking in the table of Tangents, the angle whose tangent is nearest to .83333 is 39° 48′. Hence, angle $A = 39° 48′$, nearly. Angle $B = 90° - 39° 48′ = 50° 12′$.

To find the hypotenuse, we use relation V, Art. **177**,

$$AB, \text{ hypotenuse} = \frac{\text{side opposite}}{\text{sine}} = \frac{BC}{\sin A}.$$

The sine of 39° 48′ is .64011. Hence, $AB = \frac{15}{.64011} = 23.43$.

Ans. { Angle $A = 39° 48′$. Angle $B = 50° 12′$. Side $AB = 23.43$.

EXAMPLE.—In the right-angled triangle ABC, Fig. 5, right-angled at C, $AC = .024967$ mile and $AB = .04792$ mile; to find the other parts.

SOLUTION.—According to the definition of the cosine, Art. **163.**

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{.024967}{.04792} = .52101.$$

Referring to the table, the angle whose cosine is .52101 is 58° 36′. Therefore, angle A = 58° 36′. Angle B = 90° − 58° 36′ = 31° 24′.

To find side $B\,C$, relation III, Art. **177,** is used.

Side opposite A = side adjacent × tan A, or $B\,C = A\,C \times \tan 58° 36' = .024967 \times 1.63826$ = .0409 mile.

Ans. { Angle A = 58° 36′.
Angle B = 31° 24′.
$B\,C$ = .0409 mile.

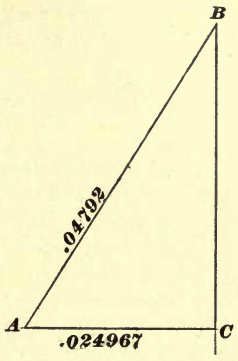

FIG. 5.

EXAMPLE.—In the right-angled triangle $A\,B\,C$, Fig. 6, right-angled at C, $A\,B$ = 308 feet and $B\,C$ = 234 feet; to find the other parts.

SOLUTION.—Sin $A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{234}{308} = .75974.$ (Art. **162.**)

The angle whose sine is .75974 = 49° 26½′, nearly, = angle A. Referring to the table of Sines, the sine of 49° 26′ is .75965, and the sine of 49° 27′ is .75984. Since the value obtained, .75974, lies nearly half way between these values, the angle lies midway between the above angles, and is 49° 26½′. Angle B = 90° − 49° 26½′ = 40° 33½′.

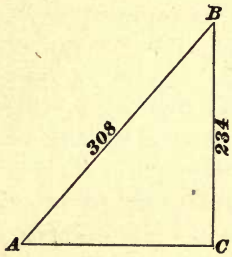

FIG. 6.

To find $A\,C$, use relation IV, Art. **177.**

Side adjacent A = side opposite × cot A, or $A\,C = 234 \times .85586 = 200.27$ feet.

Ans. { Angle A = 49° 26½′.
Angle B = 40° 33½′.
$A\,C$ = 200.27 ft.

EXAMPLES FOR PRACTICE.

181. Solve the following examples:

1. In a right triangle $A\,B\,C$, right-angled at C, the hypotenuse $A\,B$ = 40 inches and angle A = 28° 14′ 14″. Solve the triangle.

Ans. { Angle B = 61° 45′ 46″.
$A\,C$ = 35.24 in.
$B\,C$ = 18.92 in.

2. In a right triangle $A\,B\,C$, right-angled at C, the side $B\,C = 10$ feet 4 inches. If angle $A = 26°\ 59'\ 6''$, what do the other parts equal?

Ans. $\begin{cases} \text{Angle } B = 63°\ 0'\ 54''. \\ A\,B = 22 \text{ ft. } 9\frac{1}{4} \text{ in., nearly.} \\ A\,C = 20 \text{ ft. } 3\frac{1}{2} \text{ in., nearly.} \end{cases}$

3. In a right triangle $A\,B\,C$, right-angled at C, the hypotenuse $A\,B = 60$ feet and the side $A\,C = 22$ feet. Solve the triangle.

Ans. $\begin{cases} \text{Angle } A = 68°\ 29'\ 22.2''. \\ \text{Angle } B = 21°\ 30'\ 37.8''. \\ B\,C = 55.82 \text{ ft.} \end{cases}$

4. In a right triangle $A\,B\,C$, right-angled at C, side $A\,C = .364$ foot and side $B\,C = .216$ foot. Solve the triangle.

Ans. $\begin{cases} \text{Angle } A = 30°\ 41'\ 7.5''. \\ \text{Angle } B = 59°\ 18'\ 52.5''. \\ A\,B = .423 \text{ ft.} \end{cases}$

OBLIQUE-ANGLED PLANE TRIANGLES.

182. We will give here the method of solving *any* oblique-angled plane triangle, when (1) *two sides and an angle opposite one of them* are given, and (2) when *two angles and a side opposite one of them* are given. Let the student bear in mind, however, that he may use this method of solution only when he knows the general form of the triangle of which he desires the values of some of the parts. This is necessary because in some cases *two solutions* are possible, resulting in the determinations of triangles of quite different forms. We do not think it necessary to go so deeply into the explanation of this point that the student can detect the cases where two solutions are possible.

183. The solution of the triangle depends upon the following principle: *In any triangle, the sides are proportional to the sines of the opposite angles.* Thus, referring to Fig. 7, the following proportions are true:

$$a : b = \sin A : \sin B.$$
$$a : c = \sin A : \sin C.$$
$$b : c = \sin B : \sin C.$$

The method of solving the triangle is shown in the following examples:

184. Case I.—*Two sides and an angle opposite one of them are given:*

EXAMPLE.—In the triangle $A\ B\ C$, Fig. 7, having given the side $a = 1{,}686$ feet, the side $b = 960$ feet, and the angle $A = 33° 35'$; to find the angle B.

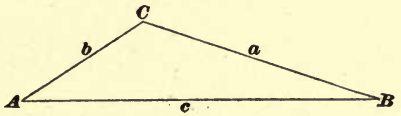

FIG. 7.

SOLUTION.—We have $a : b = \sin A : \sin B$. Substituting the known values, we have

$$1{,}686 : 960 = \sin 33° 35' : \sin B.$$

From the table of Natural Sines we get .55315 as the sine of 33° 35′. Substituting this value in the proportion and solving, we find that .31496 is the sine of the angle B, which, by consulting the table of Natural Sines, we find to correspond nearly with the angle 18° 22′. Ans.

185. Case II.—*Two angles and a side opposite one of them are given:*

EXAMPLE.—In the triangle $A\ B\ C$, given the angle $A = 33° 35'$, the angle $B = 18° 22'$, and the side $a = 1{,}686$ feet; to find the side b.

SOLUTION.—We have $\sin A : \sin B = a : b$. Substituting known values, we get

$$.55315 : .31509 = 1{,}686 : b.$$

Solving the proportion, we find $b = 960$ feet. Ans.

TABLES

OF

NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS

GIVING THE VALUES OF THE FUNCTIONS FOR ALL DEGREES AND MINUTES FROM

0° TO 90°

′	0°		1°		2°		3°		4°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.00000	1.	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.00029	1.	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.00058	1.	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.00087	1.	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.00116	1.	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.00145	1.	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.00175	1.	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.00204	1.	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.00233	1.	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.00262	1.	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.00291	1.	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.00320	.99999	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.00349	.99999	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.00378	.99999	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.00407	.99999	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.00436	.99999	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.00465	.99999	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.00495	.99999	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.00524	.99999	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.00553	.99998	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.00611	.99998	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.00640	.99998	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.00669	.99998	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.00698	.99998	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.00727	.99997	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.00756	.99997	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.00785	.99997	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.00814	.99997	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.00844	.99996	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.00902	.99996	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.00931	.99996	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.00960	.99995	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.00989	.99995	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.01018	.99995	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.01047	.99995	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.01076	.99994	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.01105	.99994	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.01134	.99994	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.01193	.99993	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.01222	.99993	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.01251	.99992	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.01280	.99992	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.01309	.99991	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.01338	.99991	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.01367	.99991	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.01396	.99990	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.01425	.99990	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.01483	.99989	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.01513	.99989	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.01542	.99988	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.01571	.99988	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.01600	.99987	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.01629	.99987	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.01658	.99986	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.01687	.99986	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.01716	.99985	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	89°		88°		87°		86°		85°		

′	5°		6°		7°		8°		9°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	60
1	.08745	.99617	.10482	.99449	.12216	.99251	.13946	.99023	.15672	.98764	59
2	.08774	.99614	.10511	.99446	.12245	.99248	.13975	.99019	.15701	.98760	58
3	.08803	.99612	.10540	.99443	.12274	.99244	.14004	.99015	.15730	.98755	57
4	.08831	.99609	.10569	.99440	.12302	.99240	.14033	.99011	.15758	.98751	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	.15787	.98746	55
6	.08889	.99604	.10626	.99434	.12360	.99233	.14090	.99002	.15816	.98741	54
7	.08918	.99602	.10655	.99431	.12389	.99230	.14119	.98998	.15845	.98737	53
8	.08947	.99599	.10684	.99428	.12418	.99226	.14148	.98994	.15873	.98732	52
9	.08976	.99596	.10713	.99424	.12447	.99222	.14177	.98990	.15902	.98728	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	.15931	.98723	50
11	.09034	.99591	.10771	.99418	.12504	.99215	.14234	.98982	.15959	.98718	49
12	.09063	.99588	.10800	.99415	.12533	.99211	.14263	.98978	.15988	.98714	48
13	.09092	.99586	.10829	.99412	.12562	.99208	.14292	.98973	.16017	.98709	47
14	.09121	.99583	.10858	.99409	.12591	.99204	.14320	.98969	.16046	.98704	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	.16074	.98700	45
16	.09179	.99578	.10916	.99402	.12649	.99197	.14378	.98961	.16103	.98695	44
17	.09208	.99575	.10945	.99399	.12678	.99193	.14407	.98957	.16132	.98690	43
18	.09237	.99572	.10973	.99396	.12706	.99189	.14436	.98953	.16160	.98686	42
19	.09266	.99570	.11002	.99393	.12735	.99186	.14464	.98948	.16189	.98681	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	.16218	.98676	40
21	.09324	.99564	.11060	.99386	.12793	.99178	.14522	.98940	.16246	.98671	39
22	.09353	.99562	.11089	.99383	.12822	.99175	.14551	.98936	.16275	.98667	38
23	.09382	.99559	.11118	.99380	.12851	.99171	.14580	.98931	.16304	.98662	37
24	.09411	.99556	.11147	.99377	.12880	.99167	.14608	.98927	.16333	.98657	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	.16361	.98652	35
26	.09469	.99551	.11205	.99370	.12937	.99160	.14666	.98919	.16390	.98648	34
27	.09498	.99548	.11234	.99367	.12966	.99156	.14695	.98914	.16419	.98643	33
28	.09527	.99545	.11263	.99364	.12995	.99152	.14723	.98910	.16447	.98638	32
29	.09556	.99542	.11291	.99360	.13024	.99148	.14752	.98906	.16476	.98633	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	.16505	.98629	30
31	.09614	.99537	.11349	.99354	.13081	.99141	.14810	.98897	.16533	.98624	29
32	.09642	.99534	.11378	.99351	.13110	.99137	.14838	.98893	.16562	.98619	28
33	.09671	.99531	.11407	.99347	.13139	.99133	.14867	.98889	.16591	.98614	27
34	.09700	.99528	.11436	.99344	.13168	.99129	.14896	.98884	.16620	.98609	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	.16648	.98604	25
36	.09758	.99523	.11494	.99337	.13226	.99122	.14954	.98876	.16677	.98600	24
37	.09787	.99520	.11523	.99334	.13254	.99118	.14982	.98871	.16706	.98595	23
38	.09816	.99517	.11552	.99331	.13283	.99114	.15011	.98867	.16734	.98590	22
39	.09845	.99514	.11580	.99327	.13312	.99110	.15040	.98863	.16763	.98585	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	.16792	.98580	20
41	.09903	.99508	.11638	.99320	.13370	.99102	.15097	.98854	.16820	.98575	19
42	.09932	.99506	.11667	.99317	.13399	.99098	.15126	.98849	.16849	.98570	18
43	.09961	.99503	.11696	.99314	.13427	.99094	.15155	.98845	.16878	.98565	17
44	.09990	.99500	.11725	.99310	.13456	.99091	.15184	.98841	.16906	.98561	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	.16935	.98556	15
46	.10048	.99494	.11783	.99303	.13514	.99083	.15241	.98832	.16964	.98551	14
47	.10077	.99491	.11812	.99300	.13543	.99079	.15270	.98827	.16992	.98546	13
48	.10106	.99488	.11840	.99297	.13572	.99075	.15299	.98823	.17021	.98541	12
49	.10135	.99485	.11869	.99293	.13600	.99071	.15327	.98818	.17050	.98536	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	.17078	.98531	10
51	.10192	.99479	.11927	.99286	.13658	.99063	.15385	.98809	.17107	.98526	9
52	.10221	.99476	.11956	.99283	.13687	.99059	.15414	.98805	.17136	.98521	8
53	.10250	.99473	.11985	.99279	.13716	.99055	.15442	.98800	.17164	.98516	7
54	.10279	.99470	.12014	.99276	.13744	.99051	.15471	.98796	.17193	.98511	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	.17222	.98506	5
56	.10337	.99464	.12071	.99269	.13802	.99043	.15529	.98787	.17250	.98501	4
57	.10366	.99461	.12100	.99265	.13831	.99039	.15557	.98782	.17279	.98496	3
58	.10395	.99458	.12129	.99262	.13860	.99035	.15586	.98778	.17308	.98491	2
59	.10424	.99455	.12158	.99258	.13889	.99031	.15615	.98773	.17336	.98486	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	.17365	.98481	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	84°		83°		82°		81°		80°		

′	10°		11°		12°		13°		14°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	60
1	.17393	.98476	.19109	.98157	.20820	.97809	.22523	.97430	.24220	.97023	59
2	.17422	.98471	.19138	.98152	.20848	.97803	.22552	.97424	.24249	.97015	58
3	.17451	.98466	.19167	.98146	.20877	.97797	.22580	.97417	.24277	.97008	57
4	.17479	.98461	.19195	.98140	.20905	.97791	.22608	.97411	.24305	.97001	56
5	.17508	.98455	.19224	.98135	.20933	.97784	.22637	.97404	.24333	.96994	55
6	.17537	.98450	.19252	.98129	.20962	.97778	.22665	.97398	.24362	.96987	54
7	.17565	.98445	.19281	.98124	.20990	.97772	.22693	.97391	.24390	.96980	53
8	.17594	.98440	.19309	.98118	.21019	.97766	.22722	.97384	.24418	.96973	52
9	.17623	.98435	.19338	.98112	.21047	.97760	.22750	.97378	.24446	.96966	51
10	.17651	.98430	.19366	.98107	.21076	.97754	.22778	.97371	.24474	.96959	50
11	.17680	.98425	.19395	.98101	.21104	.97748	.22807	.97365	.24503	.96952	49
12	.17708	.98420	.19423	.98096	.21132	.97742	.22835	.97358	.24531	.96945	48
13	.17737	.98414	.19452	.98090	.21161	.97735	.22863	.97351	.24559	.96937	47
14	.17766	.98409	.19481	.98084	.21189	.97729	.22892	.97345	.24587	.96930	46
15	.17794	.98404	.19509	.98079	.21218	.97723	.22920	.97338	.24615	.96923	45
16	.17823	.98399	.19538	.98073	.21246	.97717	.22948	.97331	.24644	.96916	44
17	.17852	.98394	.19566	.98067	.21275	.97711	.22977	.97325	.24672	.96909	43
18	.17880	.98389	.19595	.98061	.21303	.97705	.23005	.97318	.24700	.96902	42
19	.17909	.98383	.19623	.98056	.21331	.97698	.23033	.97311	.24728	.96894	41
20	.17937	.98378	.19652	.98050	.21360	.97692	.23062	.97304	.24756	.96887	40
21	.17966	.98373	.19680	.98044	.21388	.97686	.23090	.97298	.24784	.96880	39
22	.17995	.98368	.19709	.98039	.21417	.97680	.23118	.97291	.24813	.96873	38
23	.18023	.98362	.19737	.98033	.21445	.97673	.23146	.97284	.24841	.96866	37
24	.18052	.98357	.19766	.98027	.21474	.97667	.23175	.97278	.24869	.96858	36
25	.18081	.98352	.19794	.98021	.21502	.97661	.23203	.97271	.24897	.96851	35
26	.18109	.98347	.19823	.98016	.21530	.97655	.23231	.97264	.24925	.96844	34
27	.18138	.98341	.19851	.98010	.21559	.97648	.23260	.97257	.24954	.96837	33
28	.18166	.98336	.19880	.98004	.21587	.97642	.23288	.97251	.24982	.96829	32
29	.18195	.98331	.19908	.97998	.21616	.97636	.23316	.97244	.25010	.96822	31
30	.18224	.98325	.19937	.97992	.21644	.97630	.23345	.97237	.25038	.96815	30
31	.18252	.98320	.19965	.97987	.21672	.97623	.23373	.97230	.25066	.96807	29
32	.18281	.98315	.19994	.97981	.21701	.97617	.23401	.97223	.25094	.96800	28
33	.18309	.98310	.20022	.97975	.21729	.97611	.23429	.97217	.25122	.96793	27
34	.18338	.98304	.20051	.97969	.21758	.97604	.23458	.97210	.25151	.96786	26
35	.18367	.98299	.20079	.97963	.21786	.97598	.23486	.97203	.25179	.96778	25
36	.18395	.98294	.20108	.97958	.21814	.97592	.23514	.97196	.25207	.96771	24
37	.18424	.98288	.20136	.97952	.21843	.97585	.23542	.97189	.25235	.96764	23
38	.18452	.98283	.20165	.97946	.21871	.97579	.23571	.97182	.25263	.96756	22
39	.18481	.98277	.20193	.97940	.21899	.97573	.23599	.97176	.25291	.96749	21
40	.18509	.98272	.20222	.97934	.21928	.97566	.23627	.97169	.25320	.96742	20
41	.18538	.98267	.20250	.97928	.21956	.97560	.23656	.97162	.25348	.96734	19
42	.18567	.98261	.20279	.97922	.21985	.97553	.23684	.97155	.25376	.96727	18
43	.18595	.98256	.20307	.97916	.22013	.97547	.23712	.97148	.25404	.96719	17
44	.18624	.98250	.20336	.97910	.22041	.97541	.23740	.97141	.25432	.96712	16
45	.18652	.98245	.20364	.97905	.22070	.97534	.23769	.97134	.25460	.96705	15
46	.18681	.98240	.20393	.97899	.22098	.97528	.23797	.97127	.25488	.96697	14
47	.18710	.98234	.20421	.97893	.22126	.97521	.23825	.97120	.25516	.96690	13
48	.18738	.98229	.20450	.97887	.22155	.97515	.23853	.97113	.25545	.96682	12
49	.18767	.98223	.20478	.97881	.22183	.97508	.23882	.97106	.25573	.96675	11
50	.18795	.98218	.20507	.97875	.22212	.97502	.23910	.97100	.25601	.96667	10
51	.18824	.98212	.20535	.97869	.22240	.97496	.23938	.97093	.25629	.96660	9
52	.18852	.98207	.20563	.97863	.22268	.97489	.23966	.97086	.25657	.96653	8
53	.18881	.98201	.20592	.97857	.22297	.97483	.23995	.97079	.25685	.96645	7
54	.18910	.98196	.20620	.97851	.22325	.97476	.24023	.97072	.25713	.96638	6
55	.18938	.98190	.20649	.97845	.22353	.97470	.24051	.97065	.25741	.96630	5
56	.18967	.98185	.20677	.97839	.22382	.97463	.24079	.97058	.25769	.96623	4
57	.18995	.98179	.20706	.97833	.22410	.97457	.24108	.97051	.25798	.96615	3
58	.19024	.98174	.20734	.97827	.22438	.97450	.24136	.97044	.25826	.96608	2
59	.19052	.98168	.20763	.97821	.22467	.97444	.24164	.97037	.25854	.96600	1
60	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	.25882	.96593	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	79°		78°		77°		76°		75°		

′	15°		16°		17°		18°		19°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.25882	.96593	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	60
1	.25910	.96585	.27592	.96118	.29265	.95622	.30929	.95097	.32584	.94542	59
2	.25938	.96578	.27620	.96110	.29293	.95613	.30957	.95088	.32612	.94533	58
3	.25966	.96570	.27648	.96102	.29321	.95605	.30985	.95079	.32639	.94523	57
4	.25994	.96562	.27676	.96094	.29348	.95596	.31012	.95070	.32667	.94514	56
5	.26022	.96555	.27704	.96086	.29376	.95588	.31040	.95061	.32694	.94504	55
6	.26050	.96547	.27731	.96078	.29404	.95579	.31068	.95052	.32722	.94495	54
7	.26079	.96540	.27759	.96070	.29432	.95571	.31095	.95043	.32749	.94485	53
8	.26107	.96532	.27787	.96062	.29460	.95562	.31123	.95033	.32777	.94476	52
9	.26135	.96524	.27815	.96054	.29487	.95554	.31151	.95024	.32804	.94466	51
10	.26163	.96517	.27843	.96046	.29515	.95545	.31178	.95015	.32832	.94457	50
11	.26191	.96509	.27871	.96037	.29543	.95536	.31206	.95006	.32859	.94447	49
12	.26219	.96502	.27899	.96029	.29571	.95528	.31233	.94997	.32887	.94438	48
13	.26247	.96494	.27927	.96021	.29599	.95519	.31261	.94988	.32914	.94428	47
14	.26275	.96486	.27955	.96013	.29626	.95511	.31289	.94979	.32942	.94418	46
15	.26303	.96479	.27983	.96005	.29654	.95502	.31316	.94970	.32969	.94409	45
16	.26331	.96471	.28011	.95997	.29682	.95493	.31344	.94961	.32997	.94399	44
17	.26359	.96463	.28039	.95989	.29710	.95485	.31372	.94952	.33024	.94390	43
18	.26387	.96456	.28067	.95981	.29737	.95476	.31399	.94943	.33051	.94380	42
19	.26415	.96448	.28095	.95972	.29765	.95467	.31427	.94933	.33079	.94370	41
20	.26443	.96440	.28123	.95964	.29793	.95459	.31454	.94924	.33106	.94361	40
21	.26471	.96433	.28150	.95956	.29821	.95450	.31482	.94915	.33134	.94351	39
22	.26500	.96425	.28178	.95948	.29849	.95441	.31510	.94906	.33161	.94342	38
23	.26528	.96417	.28206	.95940	.29876	.95433	.31537	.94897	.33189	.94332	37
24	.26556	.96410	.28234	.95931	.29904	.95424	.31565	.94888	.33216	.94322	36
25	.26584	.96402	.28262	.95923	.29932	.95415	.31593	.94878	.33244	.94313	35
26	.26612	.96394	.28290	.95915	.29960	.95407	.31620	.94869	.33271	.94303	34
27	.26640	.96386	.28318	.95907	.29987	.95398	.31648	.94860	.33298	.94293	33
28	.26668	.96379	.28346	.95898	.30015	.95389	.31675	.94851	.33326	.94284	32
29	.26696	.96371	.28374	.95890	.30043	.95380	.31703	.94842	.33353	.94274	31
30	.26724	.96363	.28402	.95882	.30071	.95372	.31730	.94832	.33381	.94264	30
31	.26752	.96355	.28429	.95874	.30098	.95363	.31758	.94823	.33408	.94254	29
32	.26780	.96347	.28457	.95865	.30126	.95354	.31786	.94814	.33436	.94245	28
33	.26808	.96340	.28485	.95857	.30154	.95345	.31813	.94805	.33463	.94235	27
34	.26836	.96332	.28513	.95849	.30182	.95337	.31841	.94795	.33490	.94225	26
35	.26864	.96324	.28541	.95841	.30209	.95328	.31868	.94786	.33518	.94215	25
36	.26892	.96316	.28569	.95832	.30237	.95319	.31896	.94777	.33545	.94206	24
37	.26920	.96308	.28597	.95824	.30265	.95310	.31923	.94768	.33573	.94196	23
38	.26948	.96301	.28625	.95816	.30292	.95301	.31951	.94758	.33600	.94186	22
39	.26976	.96293	.28652	.95807	.30320	.95293	.31979	.94749	.33627	.94176	21
40	.27004	.96285	.28680	.95799	.30348	.95284	.32006	.94740	.33655	.94167	20
41	.27032	.96277	.28708	.95791	.30376	.95275	.32034	.94730	.33682	.94157	19
42	.27060	.96269	.28736	.95782	.30403	.95266	.32061	.94721	.33710	.94147	18
43	.27088	.96261	.28764	.95774	.30431	.95257	.32089	.94712	.33737	.94137	17
44	.27116	.96253	.28792	.95766	.30459	.95248	.32116	.94702	.33764	.94127	16
45	.27144	.96246	.28820	.95757	.30486	.95240	.32144	.94693	.33792	.94118	15
46	.27172	.96238	.28847	.95749	.30514	.95231	.32171	.94684	.33819	.94108	14
47	.27200	.96230	.28875	.95740	.30542	.95222	.32199	.94674	.33846	.94098	13
48	.27228	.96222	.28903	.95732	.30570	.95213	.32227	.94665	.33874	.94088	12
49	.27256	.96214	.28931	.95724	.30597	.95204	.32254	.94656	.33901	.94078	11
50	.27284	.96206	.28959	.95715	.30625	.95195	.32282	.94646	.33929	.94068	10
51	.27312	.96198	.28987	.95707	.30653	.95186	.32309	.94637	.33956	.94058	9
52	.27340	.96190	.29015	.95698	.30680	.95177	.32337	.94627	.33983	.94049	8
53	.27368	.96182	.29042	.95690	.30708	.95168	.32364	.94618	.34011	.94039	7
54	.27396	.96174	.29070	.95681	.30736	.95159	.32392	.94609	.34038	.94029	6
55	.27424	.96166	.29098	.95673	.30763	.95150	.32419	.94599	.34065	.94019	5
56	.27452	.96158	.29126	.95664	.30791	.95142	.32447	.94590	.34093	.94009	4
57	.27480	.96150	.29154	.95656	.30819	.95133	.32474	.94580	.34120	.93999	3
58	.27508	.96142	.29182	.95647	.30846	.95124	.32502	.94571	.34147	.93989	2
59	.27536	.96134	.29209	.95639	.30874	.95115	.32529	.94561	.34175	.93979	1
60	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	74°		73°		72°		71°		70°		

′	20°		21°		22°		23°		24°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.34202	.93969	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	60
1	.34229	.93959	.35864	.93348	.37488	.92707	.39100	.92039	.40700	.91343	59
2	.34257	.93949	.35891	.93337	.37515	.92697	.39127	.92028	.40727	.91331	58
3	.34284	.93939	.35918	.93327	.37542	.92686	.39153	.92016	.40753	.91319	57
4	.34311	.93929	.35945	.93316	.37569	.92675	.39180	.92005	.40780	.91307	56
5	.34339	.93919	.35973	.93306	.37595	.92664	.39207	.91994	.40806	.91295	55
6	.34366	.93909	.36000	.93295	.37622	.92653	.39234	.91982	.40833	.91283	54
7	.34393	.93899	.36027	.93285	.37649	.92642	.39260	.91971	.40860	.91272	53
8	.34421	.93889	.36054	.93274	.37676	.92631	.39287	.91959	.40886	.91260	52
9	.34448	.93879	.36081	.93264	.37703	.92620	.39314	.91948	.40913	.91248	51
10	.34475	.93869	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	.34503	.93859	.36135	.93243	.37757	.92598	.39367	.91925	.40966	.91224	49
12	.34530	.93849	.36162	.93232	.37784	.92587	.39394	.91914	.40992	.91212	48
13	.34557	.93839	.36190	.93222	.37811	.92576	.39421	.91902	.41019	.91200	47
14	.34584	.93829	.36217	.93211	.37838	.92565	.39448	.91891	.41045	.91188	46
15	.34612	.93819	.36244	.93201	.37865	.92554	.39474	.91879	.41072	.91176	45
16	.34639	.93809	.36271	.93190	.37892	.92543	.39501	.91868	.41098	.91164	44
17	.34666	.93799	.36298	.93180	.37919	.92532	.39528	.91856	.41125	.91152	43
18	.34694	.93789	.36325	.93169	.37946	.92521	.39555	.91845	.41151	.91140	42
19	.34721	.93779	.36352	.93159	.37973	.92510	.39581	.91833	.41178	.91128	41
20	.34748	.93769	.36379	.93148	.37999	.92497	.39608	.91822	.41204	.91116	40
21	.34775	.93759	.36406	.93137	.38026	.92488	.39635	.91810	.41231	.91104	39
22	.34803	.93748	.36434	.93127	.38053	.92477	.39661	.91799	.41257	.91092	38
23	.34830	.93738	.36461	.93116	.38080	.92466	.39688	.91787	.41284	.91080	37
24	.34857	.93728	.36488	.93106	.38107	.92455	.39715	.91775	.41310	.91068	36
25	.34884	.93718	.36515	.93095	.38134	.92444	.39741	.91764	.41337	.91056	35
26	.34912	.93708	.36542	.93084	.38161	.92432	.39768	.91752	.41363	.91044	34
27	.34939	.93698	.36569	.93074	.38188	.92421	.39795	.91741	.41390	.91032	33
28	.34966	.93688	.36596	.93063	.38215	.92410	.39822	.91729	.41416	.91020	32
29	.34993	.93677	.36623	.93052	.38241	.92399	.39848	.91718	.41443	.91008	31
30	.35021	.93667	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	.35048	.93657	.36677	.93031	.38295	.92377	.39902	.91694	.41496	.90984	29
32	.35075	.93647	.36704	.93020	.38322	.92366	.39928	.91683	.41522	.90972	28
33	.35102	.93637	.36731	.93010	.38349	.92355	.39955	.91671	.41549	.90960	27
34	.35130	.93626	.36758	.92999	.38376	.92343	.39982	.91660	.41575	.90948	26
35	.35157	.93616	.36785	.92988	.38403	.92332	.40008	.91648	.41602	.90936	25
36	.35184	.93606	.36812	.92978	.38430	.92321	.40035	.91636	.41628	.90924	24
37	.35211	.93596	.36839	.92967	.38456	.92310	.40062	.91625	.41655	.90911	23
38	.35239	.93585	.36867	.92956	.38483	.92299	.40088	.91613	.41681	.90899	22
39	.35266	.93575	.36894	.92945	.38510	.92287	.40115	.91601	.41707	.90887	21
40	.35293	.93565	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	.35320	.93555	.36948	.92924	.38564	.92265	.40168	.91578	.41760	.90863	19
42	.35347	.93544	.36975	.92913	.38591	.92254	.40195	.91566	.41787	.90851	18
43	.35375	.93534	.37002	.92902	.38617	.92243	.40221	.91555	.41813	.90839	17
44	.35402	.93524	.37029	.92892	.38644	.92231	.40248	.91543	.41840	.90826	16
45	.35429	.93514	.37056	.92881	.38671	.92220	.40275	.91531	.41866	.90814	15
46	.35456	.93503	.37083	.92870	.38698	.92209	.40301	.91519	.41892	.90802	14
47	.35484	.93493	.37110	.92859	.38725	.92198	.40328	.91508	.41919	.90790	13
48	.35511	.93483	.37137	.92849	.38752	.92186	.40355	.91496	.41945	.90778	12
49	.35538	.93472	.37164	.92838	.38778	.92175	.40381	.91484	.41972	.90766	11
50	.35565	.93462	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	.35592	.93452	.37218	.92816	.38832	.92152	.40434	.91461	.42024	.90741	9
52	.35619	.93441	.37245	.92805	.38859	.92141	.40461	.91449	.42051	.90729	8
53	.35647	.93431	.37272	.92794	.38886	.92130	.40488	.91437	.42677	.90717	7
54	.35674	.93420	.37299	.92784	.38912	.92119	.40514	.91425	.42104	.90704	6
55	.35701	.93410	.37326	.92773	.38939	.92107	.40541	.91414	.42130	.90692	5
56	.35728	.93400	.37353	.92762	.38966	.92096	.40567	.91402	.42156	.90680	4
57	.35755	.93389	.37380	.92751	.38993	.92085	.40594	.91390	.42183	.90668	3
58	.35782	.93379	.37407	.92740	.39020	.92073	.40621	.91378	.42209	.90655	2
59	.35810	.93368	.37434	.92729	.39046	.92062	.40647	.91366	.42235	.90643	1
60	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	69°		68°		67°		66°		65°		

′	25°		26°		27°		28°		29°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	60
1	.42288	.90618	.43863	.89867	.45425	.89087	.46973	.88281	.48506	.87448	59
2	.42315	.90606	.43889	.89854	.45451	.89074	.46999	.88267	.48532	.87434	58
3	.42341	.90594	.43916	.89841	.45477	.89061	.47024	.88254	.48557	.87420	57
4	.42367	.90582	.43942	.89828	.45503	.89048	.47050	.88240	.48583	.87406	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	.48608	.87391	55
6	.42420	.90557	.43994	.89803	.45554	.89021	.47101	.88213	.48634	.87377	54
7	.42446	.90545	.44020	.89790	.45580	.89008	.47127	.88199	.48659	.87363	53
8	.42473	.90532	.44046	.89777	.45606	.88995	.47153	.88185	.48684	.87349	52
9	.42499	.90520	.44072	.89764	.45632	.88981	.47178	.88172	.48710	.87335	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	.48735	.87321	50
11	.42552	.90495	.44124	.89739	.45684	.88955	.47229	.88144	.48761	.87306	49
12	.42578	.90483	.44151	.89726	.45710	.88942	.47255	.88130	.48786	.87292	48
13	.42604	.90470	.44177	.89713	.45736	.88928	.47281	.88117	.48811	.87278	47
14	.42631	.90458	.44203	.89700	.45762	.88915	.47306	.88103	.48837	.87264	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	.48862	.87250	45
16	.42683	.90433	.44255	.89674	.45813	.88888	.47358	.88075	.48888	.87235	44
17	.42709	.90421	.44281	.89662	.45839	.88875	.47383	.88062	.48913	.87221	43
18	.42736	.90408	.44307	.89649	.45865	.88862	.47409	.88048	.48938	.87207	42
19	.42762	.90396	.44333	.89636	.45891	.88848	.47434	.88034	.48964	.87193	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	.48989	.87178	40
21	.42815	.90371	.44385	.89610	.45942	.88822	.47486	.88006	.49014	.87164	39
22	.42841	.90358	.44411	.89597	.45968	.88808	.47511	.87993	.49040	.87150	38
23	.42867	.90346	.44437	.89584	.45994	.88795	.47537	.87979	.49065	.87136	37
24	.42894	.90334	.44464	.89571	.46020	.88782	.47562	.87965	.49090	.87121	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	.49116	.87107	35
26	.42946	.90309	.44516	.89545	.46072	.88755	.47614	.87937	.49141	.87093	34
27	.42972	.90296	.44542	.89532	.46097	.88741	.47639	.87923	.49166	.87079	33
28	.42999	.90284	.44568	.89519	.46123	.88728	.47665	.87909	.49192	.87064	32
29	.43025	.90271	.44594	.89506	.46149	.88715	.47690	.87896	.49217	.87050	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	.49242	.87036	30
31	.43077	.90246	.44646	.89480	.46201	.88688	.47741	.87868	.49268	.87021	29
32	.43104	.90233	.44672	.89467	.46226	.88674	.47767	.87854	.49293	.87007	28
33	.43130	.90221	.44698	.89454	.46252	.88661	.47793	.87840	.49318	.86993	27
34	.43156	.90208	.44724	.89441	.46278	.88647	.47818	.87826	.49344	.86978	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	.49369	.86964	25
36	.43209	.90183	.44776	.89415	.46330	.88620	.47869	.87798	.49394	.86949	24
37	.43235	.90171	.44802	.89402	.46355	.88607	.47895	.87784	.49419	.86935	23
38	.43261	.90158	.44828	.89389	.46381	.88593	.47920	.87770	.49445	.86921	22
39	.43287	.90146	.44854	.89376	.46407	.88580	.47946	.87756	.49470	.86906	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	.49495	.86892	20
41	.43340	.90120	.44906	.89350	.46458	.88553	.47997	.87729	.49521	.86878	19
42	.43366	.90108	.44932	.89337	.46484	.88539	.48022	.87715	.49546	.86863	18
43	.43392	.90095	.44958	.89324	.46510	.88526	.48048	.87701	.49571	.86849	17
44	.43418	.90082	.44984	.89311	.46536	.88512	.48073	.87687	.49596	.86834	16
45	.43445	.90070	.45010	.89298	.46561	.88499	.48099	.87673	.49622	.86820	15
46	.43471	.90057	.45036	.89285	.46587	.88485	.48124	.87659	.49647	.86805	14
47	.43497	.90045	.45062	.89272	.46613	.88472	.48150	.87645	.49672	.86791	13
48	.43523	.90032	.45088	.89259	.46639	.88458	.48175	.87631	.49697	.86777	12
49	.43549	.90019	.45114	.89245	.46664	.88445	.48201	.87617	.49723	.86762	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	.49748	.86748	10
51	.43602	.89994	.45166	.89219	.46716	.88417	.48252	.87589	.49773	.86733	9
52	.43628	.89981	.45192	.89206	.46742	.88404	.48277	.87575	.49798	.86719	8
53	.43654	.89968	.45218	.89193	.46767	.88390	.48303	.87561	.49824	.86704	7
54	.43680	.89956	.45243	.89180	.46793	.88377	.48328	.87546	.49849	.86690	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	.49874	.86675	5
56	.43733	.89930	.45295	.89153	.46844	.88349	.48379	.87518	.49899	.86661	4
57	.43759	.89918	.45321	.89140	.46870	.88336	.48405	.87504	.49924	.86646	3
58	.43785	.89905	.45347	.89127	.46896	.88322	.48430	.87490	.49950	.86632	2
59	.43811	.89892	.45373	.89114	.46921	.88308	.48456	.87476	.49975	.86617	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	.50000	.86603	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	64°		63°		62°		61°		60°		

′	30°		31°		32°		33°		34°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	60
1	.50025	.86588	.51529	.85702	.53017	.84789	.54488	.83851	.55943	.82887	59
2	.50050	.86573	.51554	.85687	.53041	.84774	.54513	.83835	.55968	.82871	58
3	.50076	.86559	.51579	.85672	.53066	.84759	.54537	.83819	.55992	.82855	57
4	.50101	.86544	.51604	.85657	.53091	.84743	.54561	.83804	.56016	.82839	56
5	.50126	.86530	.51628	.85642	.53115	.84728	.54586	.83788	.56040	.82822	55
6	.50151	.86515	.51653	.85627	.53140	.84712	.54610	.83772	.56064	.82806	54
7	.50176	.86501	.51678	.85612	.53164	.84697	.54635	.83756	.56088	.82790	53
8	.50201	.86486	.51703	.85597	.53189	.84681	.54659	.83740	.56112	.82773	52
9	.50227	.86471	.51728	.85582	.53214	.84666	.54683	.83724	.56136	.82757	51
10	.50252	.86457	.51753	.85567	.53238	.84650	.54708	.83708	.56160	.82741	50
11	.50277	.86442	.51778	.85551	.53263	.84635	.54732	.83692	.56184	.82724	49
12	.50302	.86427	.51803	.85536	.53288	.84619	.54756	.83676	.56208	.82708	48
13	.50327	.86413	.51828	.85521	.53312	.84604	.54781	.83660	.56232	.82692	47
14	.50352	.86398	.51852	.85506	.53337	.84588	.54805	.83645	.56256	.82675	46
15	.50377	.86384	.51877	.85491	.53361	.84573	.54829	.83629	.56280	.82659	45
16	.50403	.86369	.51902	.85476	.53386	.84557	.54854	.83613	.56305	.82643	44
17	.50428	.86354	.51927	.85461	.53411	.84542	.54878	.83597	.56329	.82626	43
18	.50453	.86340	.51952	.85446	.53435	.84526	.54902	.83581	.56353	.82610	42
19	.50478	.86325	.51977	.85431	.53460	.84511	.54927	.83565	.56377	.82593	41
20	.50503	.86310	.52002	.85416	.53484	.84495	.54951	.83549	.56401	.82577	40
21	.50528	.86295	.52026	.85401	.53509	.84480	.54975	.83533	.56425	.82561	39
22	.50553	.86281	.52051	.85385	.53534	.84464	.54999	.83517	.56449	.82544	38
23	.50578	.86266	.52076	.85370	.53558	.84448	.55024	.83501	.56473	.82528	37
24	.50603	.86251	.52101	.85355	.53583	.84433	.55048	.83485	.56497	.82511	36
25	.50628	.86237	.52126	.85340	.53607	.84417	.55072	.83469	.56521	.82495	35
26	.50654	.86222	.52151	.85325	.53632	.84402	.55097	.83453	.56545	.82478	34
27	.50679	.86207	.52175	.85310	.53656	.84386	.55121	.83437	.56569	.82462	33
28	.50704	.86192	.52200	.85294	.53681	.84370	.55145	.83421	.56593	.82446	32
29	.50729	.86178	.52225	.85279	.53705	.84355	.55169	.83405	.56617	.82429	31
30	.50754	.86163	.52250	.85264	.53730	.84339	.55194	.83389	.56641	.82413	30
31	.50779	.86148	.52275	.85249	.53754	.84324	.55218	.83373	.56665	.82396	29
32	.50804	.86133	.52299	.85234	.53779	.84308	.55242	.83356	.56689	.82380	28
33	.50829	.86119	.52324	.85218	.53804	.84292	.55266	.83340	.56713	.82363	27
34	.50854	.86104	.52349	.85203	.53828	.84277	.55291	.83324	.56736	.82347	26
35	.50879	.86089	.52374	.85188	.53853	.84261	.55315	.83308	.56760	.82330	25
36	.50904	.86074	.52399	.85173	.53877	.84245	.55339	.83292	.56784	.82314	24
37	.50929	.86059	.52423	.85157	.53902	.84230	.55363	.83276	.56808	.82297	23
38	.50954	.86045	.52448	.85142	.53926	.84214	.55388	.83260	.56832	.82281	22
39	.50979	.86030	.52473	.85127	.53951	.84198	.55412	.83244	.56856	.82264	21
40	.51004	.86015	.52498	.85112	.53975	.84182	.55436	.83228	.56880	.82248	20
41	.51029	.86000	.52522	.85096	.54000	.84167	.55460	.83212	.56904	.82231	19
42	.51054	.85985	.52547	.85081	.54024	.84151	.55484	.83195	.56928	.82214	18
43	.51079	.85970	.52572	.85066	.54049	.84135	.55509	.83179	.56952	.82198	17
44	.51104	.85956	.52597	.85051	.54073	.84120	.55533	.83163	.56976	.82181	16
45	.51129	.85941	.52621	.85035	.54097	.84104	.55557	.83147	.57000	.82165	15
46	.51154	.85926	.52646	.85020	.54122	.84088	.55581	.83131	.57024	.82148	14
47	.51179	.85911	.52671	.85005	.54146	.84072	.55605	.83115	.57047	.82132	13
48	.51204	.85896	.52696	.84989	.54171	.84057	.55630	.83098	.57071	.82115	12
49	.51229	.85881	.52720	.84974	.54195	.84041	.55654	.83082	.57095	.82098	11
50	.51254	.85866	.52745	.84959	.54220	.84025	.55678	.83066	.57119	.82082	10
51	.51279	.85851	.52770	.84943	.54244	.84009	.55702	.83050	.57143	.82065	9
52	.51304	.85836	.52794	.84928	.54269	.83994	.55726	.83034	.57167	.82048	8
53	.51329	.85821	.52819	.84913	.54293	.83978	.55750	.83017	.57191	.82032	7
54	.51354	.85806	.52844	.84897	.54317	.83962	.55775	.83001	.57215	.82015	6
55	.51379	.85792	.52869	.84882	.54342	.83946	.55799	.82985	.57238	.81999	5
56	.51404	.85777	.52893	.84866	.54366	.83930	.55823	.82969	.57262	.81982	4
57	.51429	.85762	.52918	.84851	.54391	.83915	.55847	.82953	.57286	.81965	3
58	.51454	.85747	.52943	.84836	.54415	.83899	.55871	.82936	.57310	.81949	2
59	.51479	.85732	.52967	.84820	.54440	.83883	.55895	.82920	.57334	.81932	1
60	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	.57358	.81915	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	59°		58°		57°		56°		55°		

′	35°		36°		37°		38°		39°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.57358	.81915	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	60
1	.57381	.81899	.58802	.80885	.60205	.79846	.61589	.78783	.62955	.77696	59
2	.57405	.81882	.58826	.80867	.60228	.79829	.61612	.78765	.62977	.77678	58
3	.57429	.81865	.58849	.80850	.60251	.79811	.61635	.78747	.63000	.77660	57
4	.57453	.81848	.58873	.80833	.60274	.79793	.61658	.78729	.63022	.77641	56
5	.57477	.81832	.58896	.80816	.60298	.79776	.61681	.78711	.63045	.77623	55
6	.57501	.81815	.58920	.80799	.60321	.79758	.61704	.78694	.63068	.77605	54
7	.57524	.81798	.58943	.80782	.60344	.79741	.61726	.78676	.63090	.77586	53
8	.57548	.81782	.58967	.80765	.60367	.79723	.61749	.78658	.63113	.77568	52
9	.57572	.81765	.58990	.80748	.60390	.79706	.61772	.78640	.63135	.77550	51
10	.57596	.81748	.59014	.80730	.60414	.79688	.61795	.78622	.63158	.77531	50
11	.57619	.81731	.59037	.80713	.60437	.79671	.61818	.78604	.63180	.77513	49
12	.57643	.81714	.59061	.80696	.60460	.79653	.61841	.78586	.63203	.77494	48
13	.57667	.81698	.59084	.80679	.60483	.79635	.61864	.78568	.63225	.77476	47
14	.57691	.81681	.59108	.80662	.60506	.79618	.61887	.78550	.63248	.77458	46
15	.57715	.81664	.59131	.80644	.60529	.79600	.61909	.78532	.63271	.77439	45
16	.57738	.81647	.59154	.80627	.60553	.79583	.61932	.78514	.63293	.77421	44
17	.57762	.81631	.59178	.80610	.60576	.79565	.61955	.78496	.63316	.77402	43
18	.57786	.81614	.59201	.80593	.60599	.79547	.61978	.78478	.63338	.77384	42
19	.57810	.81597	.59225	.80576	.60622	.79530	.62001	.78460	.63361	.77366	41
20	.57833	.81580	.59248	.80558	.60645	.79512	.62024	.78442	.63383	.77347	40
21	.57857	.81563	.59272	.80541	.60668	.79494	.62046	.78424	.63406	.77329	39
22	.57881	.81546	.59295	.80524	.60691	.79477	.62069	.78405	.63428	.77310	38
23	.57904	.81530	.59318	.80507	.60714	.79459	.62092	.78387	.63451	.77292	37
24	.57928	.81513	.59342	.80489	.60738	.79441	.62115	.78369	.63473	.77273	36
25	.57952	.81496	.59365	.80472	.60761	.79424	.62138	.78351	.63496	.77255	35
26	.57976	.81479	.59389	.80455	.60784	.79406	.62160	.78333	.63518	.77236	34
27	.57999	.81462	.59412	.80438	.60807	.79388	.62183	.78315	.63540	.77218	33
28	.58023	.81445	.59436	.80420	.60830	.79371	.62206	.78297	.63563	.77199	32
29	.58047	.81428	.59459	.80403	.60853	.79353	.62229	.78279	.63585	.77181	31
30	.58070	.81412	.59482	.80386	.60876	.79335	.62251	.78261	.63608	.77162	30
31	.58094	.81395	.59506	.80368	.60899	.79318	.62274	.78243	.63630	.77144	29
32	.58118	.81378	.59529	.80351	.60922	.79300	.62297	.78225	.63653	.77125	28
33	.58141	.81361	.59552	.80334	.60945	.79282	.62320	.78206	.63675	.77107	27
34	.58165	.81344	.59576	.80316	.60968	.79264	.62342	.78188	.63698	.77088	26
35	.58189	.81327	.59599	.80299	.60991	.79247	.62365	.78170	.63720	.77070	25
36	.58212	.81310	.59622	.80282	.61015	.79229	.62388	.78152	.63742	.77051	24
37	.58236	.81293	.59646	.80264	.61038	.79211	.62411	.78134	.63765	.77033	23
38	.58260	.81276	.59669	.80247	.61061	.79193	.62433	.78116	.63787	.77014	22
39	.58283	.81259	.59693	.80230	.61084	.79176	.62456	.78098	.63810	.76996	21
40	.58307	.81242	.59716	.80212	.61107	.79158	.62479	.78079	.63832	.76977	20
41	.58330	.81225	.59739	.80195	.61130	.79140	.62502	.78061	.63854	.76959	19
42	.58354	.81208	.59763	.80178	.61153	.79122	.62524	.78043	.63877	.76940	18
43	.58378	.81191	.59786	.80160	.61176	.79105	.62547	.78025	.63899	.76921	17
44	.58401	.81174	.59809	.80143	.61199	.79087	.62570	.78007	.63922	.76903	16
45	.58425	.81157	.59832	.80125	.61222	.79069	.62592	.77988	.63944	.76884	15
46	.58449	.81140	.59856	.80108	.61245	.79051	.62615	.77970	.63966	.76866	14
47	.58472	.81123	.59879	.80091	.61268	.79033	.62638	.77952	.63989	.76847	13
48	.58496	.81106	.59902	.80073	.61291	.79016	.62660	.77934	.64011	.76828	12
49	.58519	.81089	.59926	.80056	.61314	.78998	.62683	.77916	.64033	.76810	11
50	.58543	.81072	.59949	.80038	.61337	.78980	.62706	.77897	.64056	.76791	10
51	.58567	.81055	.59972	.80021	.61360	.78962	.62728	.77879	.64078	.76772	9
52	.58590	.81038	.59995	.80003	.61383	.78944	.62751	.77861	.64100	.76754	8
53	.58614	.81021	.60019	.79986	.61406	.78926	.62774	.77843	.64123	.76735	7
54	.58637	.81004	.60042	.79968	.61429	.78908	.62796	.77824	.64145	.76717	6
55	.58661	.80987	.60065	.79951	.61451	.78891	.62819	.77806	.64167	.76698	5
56	.58684	.80970	.60089	.79934	.61474	.78873	.62842	.77788	.64190	.76679	4
57	.58708	.80953	.60112	.79916	.61497	.78855	.62864	.77769	.64212	.76661	3
58	.58731	.80936	.60135	.79899	.61520	.78837	.62887	.77751	.64234	.76642	2
59	.58755	.80919	.60158	.79881	.61543	.78819	.62909	.77733	.64256	.76623	1
60	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	54°		53°		52°		51°		50°		

′	40°		41°		42°		43°		44°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.64279	.76604	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	60
1	.64301	.76586	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.64323	.76567	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.64346	.76548	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.64368	.76530	.65694	.75395	.66999	.74237	.68285	.73056	.69549	.71853	56
5	.64390	.76511	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.64412	.76492	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.64435	.76473	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.64457	.76455	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.64479	.76436	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.64501	.76417	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.64524	.76398	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.64546	.76380	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.64568	.76361	.65891	.75222	.67194	.74061	.68476	.72877	.69737	.71671	47
14	.64590	.76342	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.64612	.76323	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.64635	.76304	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.64657	.76286	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.64679	.76267	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.64701	.76248	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.64746	.76210	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.64768	.76192	.66088	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.64790	.76173	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.64812	.76154	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.64834	.76135	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.64856	.76116	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.64878	.76097	.66197	.74953	.67495	.73787	.68772	.72597	.70029	.71386	33
28	.64901	.76078	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.64923	.76059	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.64967	.76022	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.64989	.76003	.66306	.74857	.67602	.73688	.68878	.72497	.70132	.71284	28
33	.65011	.75984	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.65033	.75965	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.65055	.75946	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	25
36	.65077	.75927	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.65100	.75908	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.65122	.75889	.66436	.74741	.67730	.73570	.69004	.72377	.70257	.71162	22
39	.65144	.75870	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.65166	.75851	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.65188	.75832	.66501	.74683	.67795	.73511	.69067	.72317	.70319	.71100	19
42	.65210	.75813	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.65232	.75794	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.65254	.75775	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.65276	.75756	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.65298	.75738	.66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.65320	.75719	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.65342	.75700	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.65364	.75680	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.65386	.75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.65408	.75642	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.65430	.75623	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.65452	.75604	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.65474	.75585	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.65496	.75566	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.65518	.75547	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.65540	.75528	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.65562	.75509	.66870	.74353	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.65584	.75490	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	49°		48°		47°		46°		45°		

′	0°		1°		2°		3°		4°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.00000	Infin.	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	60
1	.00029	3437.75	.01775	56.3506	.03521	28.3994	.05270	18.9755	.07022	14.2411	59
2	.00058	1718.87	.01804	55.4415	.03550	28.1664	.05299	18.8711	.07051	14.1821	58
3	.00087	1145.92	.01833	54.5613	.03579	27.9372	.05328	18.7678	.07080	14.1235	57
4	.00116	859.436	.01862	53.7086	.03609	27.7117	.05357	18.6656	.07110	14.0655	56
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	.07139	14.0079	55
6	.00175	572.957	.01920	52.0807	.03667	27.2715	.05416	18.4645	.07168	13.9507	54
7	.00204	491.106	.01949	51.3032	.03696	27.0566	.05445	18.3655	.07197	13.8940	53
8	.00233	429.718	.01978	50.5485	.03725	26.8450	.05474	18.2677	.07227	13.8378	52
9	.00262	381.971	.02007	49.8157	.03754	26.6367	.05503	18.1708	.07256	13.7821	51
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	.07285	13.7267	50
11	.00320	312.521	.02066	48.4121	.03812	26.2296	.05562	17.9802	.07314	13.6719	49
12	.00349	286.478	.02095	47.7395	.03842	26.0307	.05591	17.8863	.07344	13.6174	48
13	.00378	264.441	.02124	47.0853	.03871	25.8348	.05620	17.7934	.07373	13.5634	47
14	.00407	245.552	.02153	46.4489	.03900	25.6418	.05649	17.7015	.07402	13.5098	46
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	.07431	13.4566	45
16	.00465	214.858	.02211	45.2261	.03958	25.2644	.05708	17.5205	.07461	13.4039	44
17	.00495	202.219	.02240	44.6386	.03987	25.0798	.05737	17.4314	.07490	13.3515	43
18	.00524	190.984	.02269	44.0661	.04016	24.8978	.05766	17.3432	.07519	13.2996	42
19	.00553	180.932	.02298	43.5081	.04046	24.7185	.05795	17.2558	.07548	13.2480	41
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	.07578	13.1969	40
21	.00611	163.700	.02357	42.4335	.04104	24.3675	.05854	17.0837	.07607	13.1461	39
22	.00640	156.259	.02386	41.9158	.04133	24.1957	.05883	16.9990	.07636	13.0958	38
23	.00669	149.465	.02415	41.4106	.04162	24.0263	.05912	16.9150	.07665	13.0458	37
24	.00698	143.237	.02444	40.9174	.04191	23.8593	.05941	16.8319	.07695	12.9962	36
25	.00727	137.507	.02473	40.4358	.04220	23.6945	.05970	16.7496	.07724	12.9469	35
26	.00756	132.219	.02502	39.9655	.04250	23.5321	.05999	16.6681	.07753	12.8981	34
27	.00785	127.321	.02531	39.5059	.04279	23.3718	.06029	16.5874	.07782	12.8496	33
28	.00815	122.774	.02560	39.0568	.04308	23.2137	.06058	16.5075	.07812	12.8014	32
29	.00844	118.540	.02589	38.6177	.04337	23.0577	.06087	16.4283	.07841	12.7536	31
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	.07870	12.7062	30
31	.00902	110.892	.02648	37.7686	.04395	22.7519	.06145	16.2722	.07899	12.6591	29
32	.00931	107.426	.02677	37.3579	.04424	22.6020	.06175	16.1952	.07929	12.6124	28
33	.00960	104.171	.02706	36.9560	.04454	22.4541	.06204	16.1190	.07958	12.5660	27
34	.00989	101.107	.02735	36.5627	.04483	22.3081	.06233	16.0435	.07987	12.5199	26
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	.08017	12.4742	25
36	.01047	95.4895	.02793	35.8006	.04541	22.0217	.06291	15.8945	.08046	12.4288	24
37	.01076	92.9085	.02822	35.4313	.04570	21.8813	.06321	15.8211	.08075	12.3838	23
38	.01105	90.4633	.02851	35.0695	.04599	21.7426	.06350	15.7483	.08104	12.3390	22
39	.01135	88.1436	.02881	34.7151	.04628	21.6056	.06379	15.6762	.08134	12.2946	21
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	.08163	12.2505	20
41	.01193	83.8435	.02939	34.0273	.04687	21.3369	.06437	15.5340	.08192	12.2067	19
42	.01222	81.8470	.02968	33.6935	.04716	21.2049	.06467	15.4638	.08221	12.1632	18
43	.01251	79.9434	.02997	33.3662	.04745	21.0747	.06496	15.3943	.08251	12.1201	17
44	.01280	78.1263	.03026	33.0452	.04774	20.9460	.06525	15.3254	.08280	12.0772	16
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	.08309	12.0346	15
46	.01338	74.7292	.03084	32.4213	.04833	20.6932	.06584	15.1893	.08339	11.9923	14
47	.01367	73.1390	.03114	32.1181	.04862	20.5691	.06613	15.1222	.08368	11.9504	13
48	.01396	71.6151	.03143	31.8205	.04891	20.4465	.06642	15.0557	.08397	11.9087	12
49	.01425	70.1533	.03172	31.5284	.04920	20.3253	.06671	14.9898	.08427	11.8673	11
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	.08456	11.8262	10
51	.01484	67.4019	.03230	30.9599	.04978	20.0872	.06730	14.8596	.08485	11.7853	9
52	.01513	66.1055	.03259	30.6833	.05007	19.9702	.06759	14.7954	.08514	11.7448	8
53	.01542	64.8580	.03288	30.4116	.05037	19.8546	.06788	14.7317	.08544	11.7045	7
54	.01571	63.6567	.03317	30.1446	.05066	19.7403	.06817	14.6685	.08573	11.6645	6
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	.08602	11.6248	5
56	.01629	61.3829	.03376	29.6245	.05124	19.5156	.06876	14.5438	.08632	11.5853	4
57	.01658	60.3058	.03405	29.3711	.05153	19.4051	.06905	14.4823	.08661	11.5461	3
58	.01687	59.2659	.03434	29.1220	.05182	19.2959	.06934	14.4212	.08690	11.5072	2
59	.01716	58.2612	.03463	28.8771	.05212	19.1879	.06963	14.3607	.08720	11.4685	1
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	.08749	11.4301	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	89°		88°		87°		86°		85°		

′	5°		6°		7°		8°		9°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.08749	11.4301	.10510	9.51436	.12278	8.14435	.14054	7.11537	.15838	6.31375	60
1	.08778	11.3919	.10540	9.48781	.12308	8.12481	.14084	7.10038	.15868	6.30189	59
2	.08807	11.3540	.10569	9.46141	.12338	8.10536	.14113	7.08546	.15898	6.29007	58
3	.08837	11.3163	.10599	9.43515	.12367	8.08600	.14143	7.07059	.15928	6.27829	57
4	.08866	11.2789	.10628	9.40904	.12397	8.06674	.14173	7.05579	.15958	6.26655	56
5	.08895	11.2417	.10657	9.38307	.12426	8.04756	.14202	7.04105	.15988	6.25486	55
6	.08925	11.2048	.10687	9.35724	.12456	8.02848	.14232	7.02637	.16017	6.24321	54
7	.08954	11.1681	.10716	9.33155	.12485	8.00948	.14262	7.01174	.16047	6.23160	53
8	.08983	11.1316	.10746	9.30599	.12515	7.99058	.14291	6.99718	.16077	6.22003	52
9	.09013	11.0954	.10775	9.28058	.12544	7.97176	.14321	6.98268	.16107	6.20851	51
10	.09042	11.0594	.10805	9.25530	.12574	7.95302	.14351	6.96823	.16137	6.19703	50
11	.09071	11.0237	.10834	9.23016	.12603	7.93438	.14381	6.95385	.16167	6.18559	49
12	.09101	10.9882	.10863	9.20516	.12633	7.91582	.14410	6.93952	.16196	6.17419	48
13	.09130	10.9529	.10893	9.18028	.12662	7.89734	.14440	6.92525	.16226	6.16283	47
14	.09159	10.9178	.10922	9.15554	.12692	7.87895	.14470	6.91104	.16256	6.15151	46
15	.09189	10.8829	.10952	9.13093	.12722	7.86064	.14499	6.89688	.16286	6.14023	45
16	.09218	10.8483	.10981	9.10646	.12751	7.84242	.14529	6.88278	.16316	6.12899	44
17	.09247	10.8139	.11011	9.08211	.12781	7.82428	.14559	6.86874	.16346	6.11779	43
18	.09277	10.7797	.11040	9.05789	.12810	7.80622	.14588	6.85475	.16376	6.10664	42
19	.09306	10.7457	.11070	9.03379	.12840	7.78825	.14618	6.84082	.16405	6.09552	41
20	.09335	10.7119	.11099	9.00983	.12869	7.77035	.14648	6.82694	.16435	6.08444	40
21	.09365	10.6783	.11128	8.98598	.12899	7.75254	.14678	6.81812	.16465	6.07340	39
22	.09394	10.6450	.11158	8.96227	.12929	7.73480	.14707	6.79936	.16495	6.06240	38
23	.09423	10.6118	.11187	8.93867	.12958	7.71715	.14737	6.78564	.16525	6.05143	37
24	.09453	10.5789	.11217	8.91520	.12988	7.69957	.14767	6.77199	.16555	6.04051	36
25	.09482	10.5462	.11246	8.89185	.13017	7.68208	.14796	6.75838	.16585	6.02962	35
26	.09511	10.5136	.11276	8.86862	.13047	7.66466	.14826	6.74483	.16615	6.01878	34
27	.09541	10.4813	.11305	8.84551	.13076	7.64732	.14856	6.73133	.16645	6.00797	33
28	.09570	10.4491	.11335	8.82252	.13106	7.63005	.14886	6.71789	.16674	5.99720	32
29	.09600	10.4172	.11364	8.79964	.13136	7.61287	.14915	6.70450	.16704	5.98646	31
30	.09629	10.3854	.11394	8.77689	.13165	7.59575	.14945	6.69116	.16734	5.97576	30
31	.09658	10.3538	.11423	8.75425	.13195	7.57872	.14975	6.67787	.16764	5.96510	29
32	.09688	10.3224	.11452	8.73172	.13224	7.56176	.15005	6.66463	.16794	5.95448	28
33	.09717	10.2913	.11482	8.70931	.13254	7.54487	.15034	6.65144	.16824	5.94390	27
34	.09746	10.2602	.11511	8.68701	.13284	7.52806	.15064	6.63831	.16854	5.93335	26
35	.09776	10.2294	.11541	8.66482	.13313	7.51132	.15094	6.62523	.16884	5.92283	25
36	.09805	10.1988	.11570	8.64275	.13343	7.49465	.15124	6.61219	.16914	5.91236	24
37	.09834	10.1683	.11600	8.62078	.13372	7.47806	.15153	6.59921	.16944	5.90191	23
38	.09864	10.1381	.11629	8.59893	.13402	7.46154	.15183	6.58627	.16974	5.89151	22
39	.09893	10.1080	.11659	8.57718	.13432	7.44509	.15213	6.57339	.17004	5.88114	21
40	.09923	10.0780	.11688	8.55555	.13461	7.42871	.15243	6.56055	.17033	5.87080	20
41	.09952	10.0483	.11718	8.53402	.13491	7.41240	.15272	6.54777	.17063	5.86051	19
42	.09981	10.0187	.11747	8.51259	.13521	7.39616	.15302	6.53503	.17093	5.85024	18
43	.10011	9.98931	.11777	8.49128	.13550	7.37999	.15332	6.52234	.17123	5.84001	17
44	.10040	9.96007	.11806	8.47007	.13580	7.36389	.15362	6.50970	.17153	5.82982	16
45	.10069	9.93101	.11836	8.44896	.13609	7.34786	.15391	6.49710	.17183	5.81966	15
46	.10099	9.90211	.11865	8.42795	.13639	7.33190	.15421	6.48456	.17213	5.80953	14
47	.10128	9.87338	.11895	8.40705	.13669	7.31600	.15451	6.47206	.17243	5.79944	13
48	.10158	9.84482	.11924	8.38625	.13698	7.30018	.15481	6.45961	.17273	5.78938	12
49	.10187	9.81641	.11954	8.36555	.13728	7.28442	.15511	6.44720	.17303	5.77936	11
50	.10216	9.78817	.11983	8.34496	.13758	7.26873	.15540	6.43484	.17333	5.76937	10
51	.10246	9.76009	.12013	8.32446	.13787	7.25310	.15570	6.42253	.17363	5.75941	9
52	.10275	9.73217	.12042	8.30406	.13817	7.23754	.15600	6.41026	.17393	5.74949	8
53	.10305	9.70441	.12072	8.28376	.13846	7.22204	.15630	6.39804	.17423	5.73960	7
54	.10334	9.67680	.12101	8.26355	.13876	7.20661	.15660	6.38587	.17453	5.72974	6
55	.10363	9.64935	.12131	8.24345	.13906	7.19125	.15689	6.37374	.17483	5.71992	5
56	.10393	9.62205	.12160	8.22344	.13935	7.17594	.15719	6.36165	.17513	5.71013	4
57	.10422	9.59490	.12190	8.20352	.13965	7.16071	.15749	6.34961	.17543	5.70037	3
58	.10452	9.56791	.12219	8.18370	.13995	7.14553	.15779	6.33761	.17573	5.69064	2
59	.10481	9.54106	.12249	8.16398	.14024	7.13042	.15809	6.32566	.17603	5.68094	1
60	.10510	9.51436	.12278	8.14435	.14054	7.11537	.15838	6.31375	.17633	5.67128	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	84°		83°		82°		81°		80°		

′	10°		11°		12°		13°		14°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.17633	5.67128	.19438	5.14455	.21256	4.70463	.23087	4.33148	.24933	4.01078	60
1	.17663	5.66165	.19468	5.13658	.21286	4.69791	.23117	4.32573	.24964	4.00582	59
2	.17693	5.65205	.19498	5.12862	.21316	4.69121	.23148	4.32001	.24995	4.00086	58
3	.17723	5.64248	.19529	5.12069	.21347	4.68452	.23179	4.31430	.25026	3.99592	57
4	.17753	5.63295	.19559	5.11279	.21377	4.67786	.23209	4.30860	.25056	3.99099	56
5	.17783	5.62344	.19589	5.10490	.21408	4.67121	.23240	4.30291	.25087	3.98607	55
6	.17813	5.61397	.19619	5.09704	.21438	4.66458	.23271	4.29724	.25118	3.98117	54
7	.17843	5.60452	.19649	5.08921	.21469	4.65797	.23301	4.29159	.25149	3.97627	53
8	.17873	5.59511	.19680	5.08139	.21499	4.65138	.23332	4.28595	.25180	3.97139	52
9	.17903	5.58573	.19710	5.07360	.21529	4.64480	.23363	4.28032	.25211	3.96651	51
10	.17933	5.57638	.19740	5.06584	.21560	4.63825	.23393	4.27471	.25242	3.96165	50
11	.17963	5.56706	.19770	5.05809	.21590	4.63171	.23424	4.26911	.25273	3.95680	49
12	.17993	5.55777	.19801	5.05037	.21621	4.62518	.23455	4.26352	.25304	3.95196	48
13	.18023	5.54851	.19831	5.04267	.21651	4.61868	.23485	4.25795	.25335	3.94713	47
14	.18053	5.53927	.19861	5.03499	.21682	4.61219	.23516	4.25239	.25366	3.94232	46
15	.18083	5.53007	.19891	5.02734	.21712	4.60572	.23547	4.24685	.25397	3.93751	45
16	.18113	5.52090	.19921	5.01971	.21743	4.59927	.23578	4.24132	.25428	3.93271	44
17	.18143	5.51176	.19952	5.01210	.21773	4.59283	.23608	4.23580	.25459	3.92793	43
18	.18173	5.50264	.19982	5.00451	.21804	4.58641	.23639	4.23030	.25490	3.92316	42
19	.18203	5.49356	.20012	4.99695	.21834	4.58001	.23670	4.22481	.25521	3.91839	41
20	.18233	5.48451	.20042	4.98940	.21864	4.57363	.23700	4.21933	.25552	3.91364	40
21	.18263	5.47548	.20073	4.98188	.21895	4.56726	.23731	4.21387	.25583	3.90890	39
22	.18293	5.46648	.20103	4.97438	.21925	4.56091	.23762	4.20842	.25614	3.90417	38
23	.18323	5.45751	.20133	4.96690	.21956	4.55458	.23793	4.20298	.25645	3.89945	37
24	.18353	5.44857	.20164	4.95945	.21986	4.54826	.23823	4.19756	.25676	3.89474	36
25	.18384	5.43966	.20194	4.95201	.22017	4.54196	.23854	4.19215	.25707	3.89004	35
26	.18414	5.43077	.20224	4.94460	.22047	4.53568	.23885	4.18675	.25738	3.88536	34
27	.18444	5.42192	.20254	4.93721	.22078	4.52941	.23916	4.18137	.25769	3.88068	33
28	.18474	5.41309	.20285	4.92984	.22108	4.52316	.23946	4.17600	.25800	3.87601	32
29	.18504	5.40429	.20315	4.92249	.22139	4.51693	.23977	4.17064	.25831	3.87136	31
30	.18534	5.39552	.20345	4.91516	.22169	4.51071	.24008	4.16530	.25862	3.86671	30
31	.18564	5.38677	.20376	4.90785	.22200	4.50451	.24039	4.15997	.25893	3.86208	29
32	.18594	5.37805	.20406	4.90056	.22231	4.49832	.24069	4.15465	.25924	3.85745	28
33	.18624	5.36936	.20436	4.89330	.22261	4.49215	.24100	4.14934	.25955	3.85284	27
34	.18654	5.36070	.20466	4.88605	.22292	4.48600	.24131	4.14405	.25986	3.84824	26
35	.18684	5.35206	.20497	4.87882	.22322	4.47986	.24162	4.13877	.26017	3.84364	25
36	.18714	5.34345	.20527	4.87162	.22353	4.47374	.24193	4.13350	.26048	3.83906	24
37	.18745	5.33487	.20557	4.86444	.22383	4.46764	.24223	4.12825	.26079	3.83449	23
38	.18775	5.32631	.20588	4.85727	.22414	4.46155	.24254	4.12301	.26110	3.82992	22
39	.18805	5.31778	.20618	4.85013	.22444	4.45548	.24285	4.11778	.26141	3.82537	21
40	.18835	5.30928	.20648	4.84300	.22475	4.44942	.24316	4.11256	.26172	3.82083	20
41	.18865	5.30080	.20679	4.83590	.22505	4.44338	.24347	4.10736	.26203	3.81630	19
42	.18895	5.29235	.20709	4.82882	.22536	4.43735	.24377	4.10216	.26235	3.81177	18
43	.18925	5.28393	.20739	4.82175	.22567	4.43134	.24408	4.09699	.26266	3.80726	17
44	.18955	5.27553	.20770	4.81471	.22597	4.42534	.24439	4.09182	.26297	3.80276	16
45	.18986	5.26715	.20800	4.80769	.22628	4.41936	.24470	4.08666	.26328	3.79827	15
46	.19016	5.25880	.20830	4.80068	.22658	4.41340	.24501	4.08152	.26359	3.79378	14
47	.19046	5.25048	.20861	4.79370	.22689	4.40745	.24532	4.07639	.26390	3.78931	13
48	.19076	5.24218	.20891	4.78673	.22719	4.40152	.24562	4.07127	.26421	3.78485	12
49	.19106	5.23391	.20921	4.77978	.22750	4.39560	.24593	4.06616	.26452	3.78040	11
50	.19136	5.22566	.20952	4.77286	.22781	4.38969	.24624	4.06107	.26483	3.77595	10
51	.19166	5.21744	.20982	4.76595	.22811	4.38381	.24655	4.05599	.26515	3.77152	9
52	.19197	5.20925	.21013	4.75906	.22842	4.37793	.24686	4.05092	.26546	3.76709	8
53	.19227	5.20107	.21043	4.75219	.22872	4.37207	.24717	4.04586	.26577	3.76268	7
54	.19257	5.19293	.21073	4.74534	.22903	4.36623	.24747	4.04081	.26608	3.75828	6
55	.19287	5.18480	.21104	4.73851	.22934	4.36040	.24778	4.03578	.26639	3.75388	5
56	.19317	5.17671	.21134	4.73170	.22964	4.35459	.24809	4.03076	.26670	3.74950	4
57	.19347	5.16863	.21164	4.72490	.22995	4.34879	.24840	4.02574	.26701	3.74512	3
58	.19378	5.16058	.21195	4.71813	.23026	4.34300	.24871	4.02074	.26733	3.74075	2
59	.19408	5.15256	.21225	4.71137	.23056	4.33723	.24902	4.01576	.26764	3.73640	1
60	.19438	5.14455	.21256	4.70463	.23087	4.33148	.24933	4.01078	.26795	3.73205	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	79°		78°		77°		76°		75°		

′	15°		16°		17°		18°		19°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.26795	3.73205	.28075	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	60
1	.26826	3.72771	.28706	3.48359	.30605	3.26745	.32524	3.07464	.34465	2.90147	59
2	.26857	3.72338	.28738	3.47977	.30637	3.26406	.32556	3.07160	.34498	2.89873	58
3	.26888	3.71907	.28769	3.47596	.30669	3.26067	.32588	3.06857	.34530	2.89600	57
4	.26920	3.71476	.28800	3.47216	.30700	3.25729	.32621	3.06554	.34563	2.89327	56
5	.26951	3.71046	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	.26982	3.70616	.28864	3.46458	.30764	3.25055	.32685	3.05950	.34628	2.88783	54
7	.27013	3.70188	.28895	3.46080	.30796	3.24719	.32717	3.05649	.34661	2.88511	53
8	.27044	3.69761	.28927	3.45703	.30828	3.24383	.32749	3.05349	.34693	2.88240	52
9	.27076	3.69335	.28958	3.45327	.30860	3.24049	.32782	3.05049	.34726	2.87970	51
10	.27107	3.68909	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2.87700	50
11	.27138	3.68485	.29021	3.44576	.30923	3.23381	.32846	3.04450	.34791	2.87430	49
12	.27169	3.68061	.29053	3.44202	.30955	3.23048	.32878	3.04152	.34824	2.87161	48
13	.27201	3.67638	.29084	3.43829	.30987	3.22715	.32911	3.03854	.34856	2.86892	47
14	.27232	3.67217	.29116	3.43456	.31019	3.22384	.32943	3.03556	.34889	2.86624	46
15	.27263	3.66796	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	.27294	3.66376	.29179	3.42713	.31083	3.21722	.33007	3.02963	.34954	2.86089	44
17	.27326	3.65957	.29210	3.42343	.31115	3.21392	.33040	3.02667	.34987	2.85822	43
18	.27357	3.65538	.29242	3.41973	.31147	3.21063	.33072	3.02372	.35020	2.85555	42
19	.27388	3.65121	.29274	3.41604	.31178	3.20734	.33104	3.02077	.35052	2.85289	41
20	.27419	3.64705	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	.27451	3.64289	.29337	3.40869	.31242	3.20079	.33169	3.01489	.35118	2.84758	39
22	.27482	3.63874	.29368	3.40502	.31274	3.19752	.33201	3.01196	.35150	2.84494	38
23	.27513	3.63461	.29400	3.40136	.31306	3.19426	.33233	3.00903	.35183	2.84229	37
24	.27545	3.63048	.29432	3.39771	.31338	3.19100	.33266	3.00611	.35216	2.83965	36
25	.27576	3.62636	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	.27607	3.62224	.29495	3.39042	.31402	3.18451	.33330	3.00028	.35281	2.83439	34
27	.27638	3.61814	.29526	3.38679	.31434	3.18127	.33363	2.99738	.35314	2.83176	33
28	.27670	3.61405	.29558	3.38317	.31466	3.17804	.33395	2.99447	.35346	2.82914	32
29	.27701	3.60996	.29590	3.37955	.31498	3.17481	.33427	2.99158	.35379	2.82653	31
30	.27732	3.60588	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	.27764	3.60181	.29653	3.37234	.31562	3.16838	.33492	2.98580	.35445	2.82130	29
32	.27795	3.59775	.29685	3.36875	.31594	3.16517	.33524	2.98292	.35477	2.81870	28
33	.27826	3.59370	.29716	3.36516	.31626	3.16197	.33557	2.98004	.35510	2.81610	27
34	.27858	3.58966	.29748	3.36158	.31658	3.15877	.33589	2.97717	.35543	2.81350	26
35	.27889	3.58562	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	.27921	3.58160	.29811	3.35443	.31722	3.15240	.33654	2.97144	.35608	2.80833	24
37	.27952	3.57758	.29843	3.35087	.31754	3.14922	.33686	2.96858	.35641	2.80574	23
38	.27983	3.57357	.29875	3.34732	.31786	3.14605	.33718	2.96573	.35674	2.80316	22
39	.28015	3.56957	.29906	3.34377	.31818	3.14288	.33751	2.96288	.35707	2.80059	21
40	.28046	3.56557	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	.28077	3.56159	.29970	3.33670	.31882	3.13656	.33816	2.95721	.35772	2.79545	19
42	.28109	3.55761	.30001	3.33317	.31914	3.13341	.33848	2.95437	.35805	2.79289	18
43	.28140	3.55364	.30033	3.32965	.31946	3.13027	.33881	2.95155	.35838	2.79033	17
44	.28172	3.54968	.30065	3.32614	.31978	3.12713	.33913	2.94872	.35871	2.78778	16
45	.28203	3.54573	.30097	3.32264	.32010	3.12400	.33945	2.94591	.35904	2.78523	15
46	.28234	3.54179	.30128	3.31914	.32042	3.12087	.33978	2.94309	.35937	2.78269	14
47	.28266	3.53785	.30160	3.31565	.32074	3.11775	.34010	2.94028	.35969	2.78014	13
48	.28297	3.53393	.30192	3.31216	.32106	3.11464	.34043	2.93748	.36002	2.77761	12
49	.28329	3.53001	.30224	3.30868	.32139	3.11153	.34075	2.93468	.36035	2.77507	11
50	.28360	3.52609	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	.28391	3.52219	.30287	3.30174	.32203	3.10532	.34140	2.92910	.36101	2.77002	9
52	.28423	3.51829	.30319	3.29829	.32235	3.10223	.34173	2.92632	.36134	2.76750	8
53	.28454	3.51441	.30351	3.29483	.32267	3.09914	.34205	2.92354	.36167	2.76498	7
54	.28486	3.51053	.30382	3.29139	.32299	3.09606	.34238	2.92076	.36199	2.76247	6
55	.28517	3.50666	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	.28549	3.50279	.30446	3.28452	.32363	3.08991	.34303	2.91523	.36265	2.75746	4
57	.28580	3.49894	.30478	3.28109	.32396	3.08685	.34335	2.91246	.36298	2.75496	3
58	.28612	3.49509	.30509	3.27767	.32428	3.08379	.34368	2.90971	.36331	2.75246	2
59	.28643	3.49125	.30541	3.27426	.32460	3.08073	.34400	2.90696	.36364	2.74997	1
60	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	74°		73°		72°		71°		70°		

′	20°		21°		22°		23°		24°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.36397	2.74748	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	60
1	.36430	2.74499	.38420	2.60283	.40436	2.47302	.42482	2.35395	.44558	2.24428	59
2	.36463	2.74251	.38453	2.60057	.40470	2.47095	.42516	2.35205	.44593	2.24252	58
3	.36496	2.74004	.38487	2.59831	.40504	2.46888	.42551	2.35015	.44627	2.24077	57
4	.36529	2.73756	.38520	2.59606	.40538	2.46682	.42585	2.34825	.44662	2.23902	56
5	.36562	2.73509	.38553	2.59381	.40572	2.46476	.42619	2.34636	.44697	2.23727	55
6	.36595	2.73263	.38587	2.59156	.40606	2.46270	.42654	2.34447	.44732	2.23553	54
7	.36628	2.73017	.38620	2.58932	.40640	2.46065	.42688	2.34258	.44767	2.23378	53
8	.36661	2.72771	.38654	2.58708	.40674	2.45860	.42722	2.34069	.44802	2.23204	52
9	.36694	2.72526	.38687	2.58484	.40707	2.45655	.42757	2.33881	.44837	2.23030	51
10	.36727	2.72281	.38721	2.58261	.40741	2.45451	.42791	2.33693	.44872	2.22857	50
11	.36760	2.72036	.38754	2.58038	.40775	2.45246	.42826	2.33505	.44907	2.22683	49
12	.36793	2.71792	.38787	2.57815	.40809	2.45043	.42860	2.33317	.44942	2.22510	48
13	.36826	2.71548	.38821	2.57593	.40843	2.44839	.42894	2.33130	.44977	2.22337	47
14	.36859	2.71305	.38854	2.57371	.40877	2.44636	.42929	2.32943	.45012	2.22164	46
15	.36892	2.71062	.38888	2.57150	.40911	2.44433	.42963	2.32756	.45047	2.21992	45
16	.36925	2.70819	.38921	2.56928	.40945	2.44230	.42998	2.32570	.45082	2.21819	44
17	.36958	2.70577	.38955	2.56707	.40979	2.44027	.43032	2.32383	.45117	2.21647	43
18	.36991	2.70335	.38988	2.56487	.41013	2.43825	.43067	2.32197	.45152	2.21475	42
19	.37024	2.70094	.39022	2.56266	.41047	2.43623	.43101	2.32012	.45187	2.21304	41
20	.37057	2.69853	.39055	2.56046	.41081	2.43422	.43136	2.31826	.45222	2.21132	40
21	.37090	2.69612	.39089	2.55827	.41115	2.43220	.43170	2.31641	.45257	2.20961	39
22	.37123	2.69371	.39122	2.55608	.41149	2.43019	.43205	2.31456	.45292	2.20790	38
23	.37157	2.69131	.39156	2.55389	.41183	2.42819	.43230	2.31271	.45327	2.20619	37
24	.37190	2.68892	.39190	2.55170	.41217	2.42618	.43274	2.31086	.45362	2.20449	36
25	.37223	2.68653	.39223	2.54952	.41251	2.42418	.43308	2.30902	.45397	2.20278	35
26	.37256	2.68414	.39257	2.54734	.41285	2.42218	.43343	2.30718	.45432	2.20108	34
27	.37289	2.68175	.39290	2.54516	.41319	2.42019	.43378	2.30534	.45467	2.19938	33
28	.37322	2.67937	.39324	2.54299	.41353	2.41819	.43412	2.30351	.45502	2.19769	32
29	.37355	2.67700	.39357	2.54082	.41387	2.41620	.43447	2.30167	.45538	2.19599	31
30	.37388	2.67462	.39391	2.53865	.41421	2.41421	.43481	2.29984	.45573	2.19430	30
31	.37422	2.67225	.39425	2.53648	.41455	2.41223	.43516	2.29801	.45608	2.19261	29
32	.37455	2.66989	.39458	2.53432	.41490	2.41025	.43550	2.29619	.45643	2.19092	28
33	.37488	2.66752	.39492	2.53217	.41524	2.40827	.43585	2.29437	.45678	2.18923	27
34	.37521	2.66516	.39526	2.53001	.41558	2.40629	.43620	2.29254	.45713	2.18755	26
35	.37554	2.66281	.39559	2.52786	.41592	2.40432	.43654	2.29073	.45748	2.18587	25
36	.37588	2.66046	.39593	2.52571	.41626	2.40235	.43689	2.28891	.45784	2.18419	24
37	.37621	2.65811	.39626	2.52357	.41660	2.40038	.43724	2.28710	.45819	2.18251	23
38	.37654	2.65576	.39660	2.52142	.41694	2.39841	.43758	2.28528	.45854	2.18084	22
39	.37687	2.65342	.39694	2.51929	.41728	2.39645	.43793	2.28348	.45889	2.17916	21
40	.37720	2.65109	.39727	2.51715	.41763	2.39449	.43828	2.28167	.45924	2.17749	20
41	.37754	2.64875	.39761	2.51502	.41797	2.39253	.43862	2.27987	.45960	2.17582	19
42	.37787	2.64642	.39795	2.51289	.41831	2.39058	.43897	2.27806	.45995	2.17416	18
43	.37820	2.64410	.39829	2.51076	.41865	2.38863	.43932	2.27626	.46030	2.17249	17
44	.37853	2.64177	.39862	2.50864	.41899	2.38668	.43966	2.27447	.46065	2.17083	16
45	.37887	2.63945	.39896	2.50652	.41933	2.38473	.44001	2.27267	.46101	2.16917	15
46	.37920	2.63714	.39930	2.50440	.41968	2.38279	.44036	2.27088	.46136	2.16751	14
47	.37953	2.63483	.39963	2.50229	.42002	2.38084	.44071	2.26909	.46171	2.16585	13
48	.37986	2.63252	.39997	2.50018	.42036	2.37891	.44105	2.26730	.46206	2.16420	12
49	.38020	2.63021	.40031	2.49807	.42070	2.37697	.44140	2.26552	.46242	2.16255	11
50	.38053	2.62791	.40065	2.49597	.42105	2.37504	.44175	2.26374	.46277	2.16090	10
51	.38086	2.62561	.40098	2.49386	.42139	2.37311	.44210	2.26196	.46312	2.15925	9
52	.38120	2.62332	.40132	2.49177	.42173	2.37118	.44244	2.26018	.46348	2.15760	8
53	.38153	2.62103	.40166	2.48967	.42207	2.36925	.44279	2.25840	.46383	2.15596	7
54	.38186	2.61874	.40200	2.48758	.42242	2.36733	.44314	2.25663	.46418	2.15432	6
55	.38220	2.61646	.40234	2.48549	.42276	2.36541	.44349	2.25486	.46454	2.15268	5
56	.38253	2.61418	.40267	2.48340	.42310	2.36349	.44384	2.25309	.46489	2.15104	4
57	.38286	2.61190	.40301	2.48132	.42345	2.36158	.44418	2.25132	.46525	2.14940	3
58	.38320	2.60963	.40335	2.47924	.42379	2.35967	.44453	2.24956	.46560	2.14777	2
59	.38353	2.60736	.40369	2.47716	.42413	2.35776	.44488	2.24780	.46595	2.14614	1
60	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	.46631	2.14451	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	69°		68°		67°		66°		65°		

′	25°		26°		27°		28°		29°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	.55431	1.80405	60
1	.46666	2.14288	.48809	2.04879	.50989	1.96120	.53208	1.87941	.55469	1.80281	59
2	.46702	2.14125	.48845	2.04728	.51026	1.95979	.53246	1.87809	.55507	1.80158	58
3	.46737	2.13963	.48881	2.04577	.51063	1.95838	.53283	1.87677	.55545	1.80034	57
4	.46772	2.13801	.48917	2.04426	.51099	1.95698	.53320	1.87546	.55583	1.79911	56
5	.46808	2.13639	.48953	2.04276	.51136	1.95557	.53358	1.87415	.55621	1.79788	55
6	.46843	2.13477	.48989	2.04125	.51173	1.95417	.53395	1.87283	.55659	1.79665	54
7	.46879	2.13316	.49026	2.03975	.51209	1.95277	.53432	1.87152	.55697	1.79542	53
8	.46914	2.13154	.49062	2.03825	.51246	1.95137	.53470	1.87021	.55736	1.79419	52
9	.46950	2.12993	.49098	2.03675	.51283	1.94997	.53507	1.86891	.55774	1.79296	51
10	.46985	2.12832	.49134	2.03526	.51319	1.94858	.53545	1.86760	.55812	1.79174	50
11	.47021	2.12671	.49170	2.03376	.51356	1.94718	.53582	1.86630	.55850	1.79051	49
12	.47056	2.12511	.49206	2.03227	.51393	1.94579	.53620	1.86499	.55888	1.78929	48
13	.47092	2.12350	.49242	2.03078	.51430	1.94440	.53657	1.86369	.55926	1.78807	47
14	.47128	2.12190	.49278	2.02929	.51467	1.94301	.55694	1.86239	.55964	1.78685	46
15	.47163	2.12030	.49315	2.02780	.51503	1.94162	.53732	1.86109	.56003	1.78563	45
16	.47199	2.11871	.49351	2.02631	.51540	1.94023	.53769	1.85979	.56041	1.78441	44
17	.47234	2.11711	.49387	2.02483	.51577	1.93885	.53807	1.85850	.56079	1.78319	43
18	.47270	2.11552	.49423	2.02335	.51614	1.93746	.53844	1.85720	.56117	1.78198	42
19	.47305	2.11392	.49459	2.02187	.51651	1.93608	.53882	1.85591	.56156	1.78077	41
20	.47341	2.11233	.49495	2.02039	.51688	1.93470	.53920	1.85462	.56194	1.77955	40
21	.47377	2.11075	.49532	2.01891	.51724	1.93332	.53957	1.85333	.56232	1.77834	39
22	.47412	2.10916	.49568	2.01743	.51761	1.93195	.53995	1.85204	.56270	1.77713	38
23	.47448	2.10758	.49604	2.01596	.51798	1.93057	.54032	1.85075	.56309	1.77592	37
24	.47483	2.10600	.49640	2.01449	.51835	1.92920	.54070	1.84946	.56347	1.77471	36
25	.47519	2.10442	.49677	2.01302	.51872	1.92782	.54107	1.84818	.56385	1.77351	35
26	.47555	2.10284	.49713	2.01155	.51909	1.92645	.54145	1.84689	.56424	1.77230	34
27	.47590	2.10126	.49749	2.01008	.51946	1.92508	.54183	1.84561	.56462	1.77110	33
28	.47626	2.09969	.49786	2.00862	.51983	1.92371	.54220	1.84433	.56501	1.76990	32
29	.47662	2.09811	.49822	2.00715	.52020	1.92235	.54258	1.84305	.56539	1.76869	31
30	.47698	2.09654	.49858	2.00569	.52057	1.92098	.54296	1.84177	.56577	1.76749	30
31	.47733	2.09498	.49894	2.00423	.52094	1.91962	.54333	1.84049	.56616	1.76629	29
32	.47769	2.09341	.49931	2.00277	.52131	1.91826	.54371	1.83922	.56654	1.76510	28
33	.47805	2.09184	.49967	2.00131	.52168	1.91690	.54409	1.83794	.56693	1.76390	27
34	.47840	2.09028	.50004	1.99986	.52205	1.91554	.54446	1.83667	.56731	1.76271	26
35	.47876	2.08872	.50040	1.99841	.52242	1.91418	.54484	1.83540	.56769	1.76151	25
36	.47912	2.08716	.50076	1.99695	.52279	1.91282	.54522	1.83413	.56808	1.76032	24
37	.47948	2.08560	.50113	1.99550	.52316	1.91147	.54560	1.83286	.56846	1.75913	23
38	.47984	2.08405	.50149	1.99406	.52353	1.91012	.54597	1.83159	.56885	1.75794	22
39	.48019	2.08250	.50185	1.99261	.52390	1.90876	.54635	1.83033	.56923	1.75675	21
40	.48055	2.08094	.50222	1.99116	.52427	1.90741	.54673	1.82906	.56962	1.75556	20
41	.48091	2.07939	.50258	1.98972	.52464	1.90607	.54711	1.82780	.57000	1.75437	19
42	.48127	2.07785	.50295	1.98828	.52501	1.90472	.54748	1.82654	.57039	1.75319	18
43	.48163	2.07630	.50331	1.98684	.52538	1.90337	.54786	1.82528	.57078	1.75200	17
44	.48198	2.07476	.50368	1.98540	.52575	1.90203	.54824	1.82402	.57116	1.75082	16
45	.48234	2.07321	.50404	1.98396	.52613	1.90069	.54862	1.82276	.57155	1.74964	15
46	.48270	2.07167	.50441	1.98253	.52650	1.89935	.54900	1.82150	.57193	1.74846	14
47	.48306	2.07014	.50477	1.98110	.52687	1.89801	.54938	1.82025	.57232	1.74728	13
48	.48342	2.06860	.50514	1.97966	.52724	1.89667	.54975	1.81899	.57271	1.74610	12
49	.48378	2.06706	.50550	1.97823	.52761	1.89533	.55013	1.81774	.57309	1.74492	11
50	.48414	2.06553	.50587	1.97681	.52798	1.89400	.55051	1.81649	.57348	1.74375	10
51	.48450	2.06400	.50623	1.97538	.52836	1.89266	.55089	1.81524	.57386	1.74257	9
52	.48486	2.06247	.50660	1.97395	.52873	1.89133	.55127	1.81399	.57425	1.74140	8
53	.48521	2.06094	.50696	1.97253	.52910	1.89000	.55165	1.81274	.57464	1.74022	7
54	.48557	2.05942	.50733	1.97111	.52947	1.88867	.55203	1.81150	.57503	1.73905	6
55	.48593	2.05790	.50769	1.96969	.52985	1.88734	.55241	1.81025	.57541	1.73788	5
56	.48629	2.05637	.50806	1.96827	.53022	1.88602	.55279	1.80901	.57580	1.73671	4
57	.48665	2.05485	.50843	1.96685	.53059	1.88469	.55317	1.80777	.57619	1.73555	3
58	.48701	2.05333	.50879	1.96544	.53096	1.88337	.55355	1.80653	.57657	1.73438	2
59	.48737	2.05182	.50916	1.96402	.53134	1.88205	.55393	1.80529	.57696	1.73321	1
60	.48773	2.05030	.50953	1.96261	.53171	1.88073	.55431	1.80405	.57735	1.73205	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	64°		63°		62°		61°		60°		

′	30°		31°		32°		33°		34°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.57735	1.73205	.60086	1.66428	.62487	1.60033	.64941	1.53986	.67451	1.48256	60
1	.57774	1.73089	.60126	1.66318	.62527	1.59930	.64982	1.53888	.67493	1.48163	59
2	.57813	1.72973	.60165	1.66209	.62568	1.59826	.65024	1.53791	.67536	1.48070	58
3	.57851	1.72857	.60205	1.66099	.62608	1.59723	.65065	1.53693	.67578	1.47977	57
4	.57890	1.72741	.60245	1.65990	.62649	1.59620	.65106	1.53595	.67620	1.47885	56
5	.57929	1.72625	.60284	1.65881	.62689	1.59517	.65148	1.53497	.67663	1.47792	55
6	.57968	1.72509	.60324	1.65772	.62730	1.59414	.65189	1.53400	.67705	1.47699	54
7	.58007	1.72393	.60364	1.65663	.62770	1.59311	.65231	1.53302	.67748	1.47607	53
8	.58046	1.72278	.60403	1.65554	.62811	1.59208	.65272	1.53205	.67790	1.47514	52
9	.58085	1.72163	.60443	1.65445	.62852	1.59105	.65314	1.53107	.67832	1.47422	51
10	.58124	1.72047	.60483	1.65337	.62892	1.59002	.65355	1.53010	.67875	1.47330	50
11	.58162	1.71932	.60522	1.65228	.62933	1.58900	.65397	1.52913	.67917	1.47238	49
12	.58201	1.71817	.60562	1.65120	.62973	1.58797	.65438	1.52816	.67960	1.47146	48
13	.58240	1.71702	.60602	1.65011	.63014	1.58695	.65480	1.52719	.68002	1.47053	47
14	.58279	1.71588	.60642	1.64903	.63055	1.58593	.65521	1.52622	.68045	1.46962	46
15	.58318	1.71473	.60681	1.64795	.63095	1.58490	.65563	1.52525	.68088	1.46870	45
16	.58357	1.71358	.60721	1.64687	.63136	1.58388	.65604	1.52429	.68130	1.46778	44
17	.58396	1.71244	.60761	1.64579	.63177	1.58286	.65646	1.52332	.68173	1.46686	43
18	.58435	1.71129	.60801	1.64471	.63217	1.58184	.65688	1.52235	.68215	1.46595	42
19	.58474	1.71015	.60841	1.64363	.63258	1.58083	.65729	1.52139	.68258	1.46503	41
20	.58513	1.70901	.60881	1.64256	.63299	1.57981	.65771	1.52043	.68301	1.46411	40
21	.58552	1.70787	.60921	1.64148	.63340	1.57879	.65813	1.51946	.68343	1.46320	39
22	.58591	1.70673	.60960	1.64041	.63380	1.57778	.65854	1.51850	.68386	1.46229	38
23	.58631	1.70560	.61000	1.63934	.63421	1.57676	.65896	1.51754	.68429	1.46137	37
24	.58670	1.70446	.61040	1.63826	.63462	1.57575	.65938	1.51658	.68471	1.46046	36
25	.58709	1.70332	.61080	1.63719	.63503	1.57474	.65980	1.51562	.68514	1.45955	35
26	.58748	1.70219	.61120	1.63612	.63544	1.57372	.66021	1.51466	.68557	1.45864	34
27	.58787	1.70106	.61160	1.63505	.63584	1.57271	.66063	1.51370	.68600	1.45773	33
28	.58826	1.69992	.61200	1.63398	.63625	1.57170	.66105	1.51275	.68642	1.45682	32
29	.58865	1.69879	.61240	1.63292	.63666	1.57069	.66147	1.51179	.68685	1.45592	31
30	.58905	1.69766	.61280	1.63185	.63707	1.56969	.66189	1.51084	.68728	1.45501	30
31	.58944	1.69653	.61320	1.63079	.63748	1.56868	.66230	1.50988	.68771	1.45410	29
32	.58983	1.69541	.61360	1.62972	.63789	1.56767	.66272	1.50893	.68814	1.45320	28
33	.59022	1.69428	.61400	1.62866	.63830	1.56667	.66314	1.50797	.68857	1.45229	27
34	.59061	1.69316	.61440	1.62760	.63871	1.56566	.66356	1.50702	.68900	1.45139	26
35	.59101	1.69203	.61480	1.62654	.63912	1.56466	.66398	1.50607	.68942	1.45049	25
36	.59149	1.69091	.61520	1.62548	.63953	1.56366	.66440	1.50512	.68985	1.44958	24
37	.59179	1.68979	.61561	1.62442	.63994	1.56265	.66482	1.50417	.69028	1.44868	23
38	.59218	1.68866	.61601	1.62336	.64035	1.56165	.66524	1.50322	.69071	1.44778	22
39	.59258	1.68754	.61641	1.62230	.64076	1.56065	.66566	1.50228	.69114	1.44688	21
40	.59297	1.68643	.61681	1.62125	.64117	1.55966	.66608	1.50133	.69157	1.44598	20
41	.59336	1.68531	.61721	1.62019	.64158	1.55866	.66650	1.50038	.69200	1.44508	19
42	.59376	1.68419	.61761	1.61914	.64199	1.55766	.66692	1.49944	.69243	1.44418	18
43	.59415	1.68308	.61801	1.61808	.64240	1.55666	.66734	1.49849	.69286	1.44329	17
44	.59454	1.68196	.61842	1.61703	.64281	1.55567	.66776	1.49755	.69329	1.44239	16
45	.59494	1.68085	.61882	1.61598	.64322	1.55467	.66818	1.49661	.69372	1.44149	15
46	.59533	1.67974	.61922	1.61493	.64363	1.55368	.66860	1.49566	.69416	1.44060	14
47	.59573	1.67863	.61962	1.61388	.64404	1.55269	.66902	1.49472	.69459	1.43970	13
48	.59612	1.67752	.62003	1.61283	.64446	1.55170	.66944	1.49378	.69502	1.43881	12
49	.59651	1.67641	.62043	1.61179	.64487	1.55071	.66986	1.49284	.69545	1.43792	11
50	.59691	1.67530	.62083	1.61074	.64528	1.54972	.67028	1.49190	.69588	1.43703	10
51	.59730	1.67419	.62124	1.60970	.64569	1.54873	.67071	1.49097	.69631	1.43614	9
52	.59770	1.67309	.62164	1.60865	.64610	1.54774	.67113	1.49003	.69675	1.43525	8
53	.59809	1.67198	.62204	1.60761	.64652	1.54675	.67155	1.48909	.69718	1.43436	7
54	.59849	1.67088	.62245	1.60657	.64693	1.54576	.67197	1.48816	.69761	1.43347	6
55	.59888	1.66978	.62285	1.60553	.64734	1.54478	.67239	1.48722	.69804	1.43258	5
56	.59928	1.66867	.62325	1.60449	.64775	1.54379	.67282	1.48629	.69847	1.43169	4
57	.59967	1.66757	.62366	1.60345	.64817	1.54281	.67324	1.48536	.69891	1.43080	3
58	.60007	1.66647	.62406	1.60241	.64858	1.54183	.67366	1.48442	.69934	1.42992	2
59	.60046	1.66538	.62446	1.60137	.64899	1.54085	.67409	1.48349	.69977	1.42903	1
60	.60086	1.66428	.62487	1.60033	.64941	1.53986	.67451	1.48256	.70021	1.42815	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	59°		58°		57°		56°		55°		

′	35°		36°		37°		38°		39°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.70021	1.42815	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	60
1	.70064	1.42726	.72699	1.37554	.75401	1.32624	.78175	1.27917	.81027	1.23416	59
2	.70107	1.42638	.72743	1.37470	.75447	1.32544	.78222	1.27841	.81075	1.23343	58
3	.70151	1.42550	.72788	1.37386	.75492	1.32464	.78269	1.27764	.81123	1.23270	57
4	.70194	1.42462	.72832	1.37302	.75538	1.32384	.78316	1.27688	.81171	1.23196	56
5	.70238	1.42374	.72877	1.37218	.75584	1.32304	.78363	1.27611	.81220	1.23123	55
6	.70281	1.42286	.72921	1.37134	.75629	1.32224	.78410	1.27535	.81268	1.23050	54
7	.70325	1.42198	.72966	1.37050	.75675	1.32144	.78457	1.27458	.81316	1.22977	53
8	.70368	1.42110	.73010	1.36967	.75721	1.32064	.78504	1.27382	.81364	1.22904	52
9	.70412	1.42022	.73055	1.36883	.75767	1.31984	.78551	1.27306	.81413	1.22831	51
10	.70455	1.41934	.73100	1.36800	.75812	1.31904	.78598	1.27230	.81461	1.22758	50
11	.70499	1.41847	.73144	1.36716	.75858	1.31825	.78645	1.27153	.81510	1.22685	49
12	.70542	1.41759	.73189	1.36633	.75904	1.31745	.78692	1.27077	.81558	1.22612	48
13	.70586	1.41672	.73234	1.36549	.75950	1.31666	.78739	1.27001	.81606	1.22539	47
14	.70629	1.41584	.73278	1.36466	.75996	1.31586	.78786	1.26925	.81655	1.22467	46
15	.70673	1.41497	.73323	1.36383	.76042	1.31507	.78834	1.26849	.81703	1.22394	45
16	.70717	1.41409	.73368	1.36300	.76088	1.31427	.78881	1.26774	.81752	1.22321	44
17	.70760	1.41322	.73413	1.36217	.76134	1.31348	.78928	1.26698	.81800	1.22249	43
18	.70804	1.41235	.73457	1.36134	.76180	1.31269	.78975	1.26622	.81849	1.22176	42
19	.70848	1.41148	.73502	1.36051	.76226	1.31190	.79022	1.26546	.81898	1.22104	41
20	.70891	1.41061	.73547	1.35968	.76272	1.31110	.79070	1.26471	.81946	1.22031	40
21	.70935	1.40974	.73592	1.35885	.76318	1.31031	.79117	1.26395	.81995	1.21959	39
22	.70979	1.40887	.73637	1.35802	.76364	1.30952	.79164	1.26319	.82044	1.21886	38
23	.71023	1.40800	.73681	1.35719	.76410	1.30873	.79212	1.26244	.82092	1.21814	37
24	.71066	1.40714	.73726	1.35637	.76456	1.30795	.79259	1.26169	.82141	1.21742	36
25	.71110	1.40627	.73771	1.35554	.76502	1.30716	.79306	1.26093	.82190	1.21670	35
26	.71154	1.40540	.73816	1.35472	.76548	1.30637	.79354	1.26018	.82238	1.21598	34
27	.71198	1.40454	.73861	1.35389	.76594	1.30558	.79401	1.25943	.82287	1.21526	33
28	.71242	1.40367	.73906	1.35307	.76640	1.30480	.79449	1.25867	.82336	1.21454	32
29	.71285	1.40281	.73951	1.35224	.76686	1.30401	.79496	1.25792	.82385	1.21382	31
30	.71329	1.40195	.73996	1.35142	.76733	1.30323	.79544	1.25717	.82434	1.21310	30
31	.71373	1.40109	.74041	1.35060	.76779	1.30244	.79591	1.25642	.82483	1.21238	29
32	.71417	1.40022	.74086	1.34978	.76825	1.30166	.79639	1.25567	.82531	1.21166	28
33	.71461	1.39936	.74131	1.34896	.76871	1.30087	.79686	1.25492	.82580	1.21094	27
34	.71505	1.39850	.74176	1.34814	.76918	1.30009	.79734	1.25417	.82629	1.21023	26
35	.71549	1.39764	.74221	1.34732	.76964	1.29931	.79781	1.25343	.82678	1.20951	25
36	.71593	1.39679	.74267	1.34650	.77010	1.29853	.79829	1.25268	.82727	1.20879	24
37	.71637	1.39593	.74312	1.34568	.77057	1.29775	.79877	1.25193	.82776	1.20808	23
38	.71681	1.39507	.74357	1.34487	.77103	1.29696	.79924	1.25118	.82825	1.20736	22
39	.71725	1.39421	.74402	1.34405	.77149	1.29618	.79972	1.25044	.82874	1.20665	21
40	.71769	1.39336	.74447	1.34323	.77196	1.29541	.80020	1.24969	.82923	1.20593	20
41	.71813	1.39250	.74492	1.34242	.77242	1.29463	.80067	1.24895	.82972	1.20522	19
42	.71857	1.39165	.74538	1.34160	.77289	1.29385	.80115	1.24820	.83022	1.20451	18
43	.71901	1.39079	.74583	1.34079	.77335	1.29307	.80163	1.24746	.83071	1.20379	17
44	.71946	1.38994	.74628	1.33998	.77382	1.29229	.80211	1.24672	.83120	1.20308	16
45	.71990	1.38909	.74674	1.33916	.77428	1.29152	.80258	1.24597	.83169	1.20237	15
46	.72034	1.38824	.74719	1.33835	.77475	1.29074	.80306	1.24523	.83218	1.20166	14
47	.72078	1.38738	.74764	1.33754	.77521	1.28997	.80354	1.24449	.83268	1.20095	13
48	.72122	1.38653	.74810	1.33673	.77568	1.28919	.80402	1.24375	.83317	1.20024	12
49	.72167	1.38568	.74855	1.33592	.77615	1.28842	.80450	1.24301	.83366	1.19953	11
50	.72211	1.38484	.74900	1.33511	.77661	1.28764	.80498	1.24227	.83415	1.19882	10
51	.72255	1.38399	.74946	1.33430	.77708	1.28687	.80546	1.24153	.83465	1.19811	9
52	.72299	1.38314	.74991	1.33349	.77754	1.28610	.80594	1.24079	.83514	1.19740	8
53	.72344	1.38229	.75037	1.33268	.77801	1.28533	.80642	1.24005	.83564	1.19669	7
54	.72388	1.38145	.75082	1.33187	.77848	1.28456	.80690	1.23931	.83613	1.19599	6
55	.72432	1.38060	.75128	1.33107	.77895	1.28379	.80738	1.23858	.83662	1.19528	5
56	.72477	1.37976	.75173	1.33026	.77941	1.28302	.80786	1.23784	.83712	1.19457	4
57	.72521	1.37891	.75219	1.32946	.77988	1.28225	.80834	1.23710	.83761	1.19387	3
58	.72565	1.37807	.75264	1.32865	.78035	1.28148	.80882	1.23637	.83811	1.19316	2
59	.72610	1.37722	.75310	1.32785	.78082	1.28071	.80930	1.23563	.83860	1.19246	1
60	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	.83910	1.19175	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	54°		53°		52°		51°		50°		

′	40°		41°		42°		43°		44°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.83910	1.19175	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	60
1	.83960	1.19105	.86980	1.14969	.90093	1.10996	.93306	1.07174	.96625	1.03493	59
2	.84009	1.19035	.87031	1.14902	.90146	1.10931	.93360	1.07112	.96681	1.03433	58
3	.84059	1.18964	.87082	1.14834	.90199	1.10867	.93415	1.07049	.96738	1.03372	57
4	.84108	1.18894	.87133	1.14767	.90251	1.10802	.93469	1.06987	.96794	1.03312	56
5	.84158	1.18824	.87184	1.14699	.90304	1.10737	.93524	1.06925	.96850	1.03252	55
6	.84208	1.18754	.87236	1.14632	.90357	1.10672	.93578	1.06862	.96907	1.03192	54
7	.84258	1.18684	.87287	1.14565	.90410	1.10607	.93633	1.06800	.96963	1.03132	53
8	.84307	1.18614	.87338	1.14498	.90463	1.10543	.93688	1.06738	.97020	1.03072	52
9	.84357	1.18544	.87389	1.14430	.90516	1.10478	.93742	1.06676	.97076	1.03012	51
10	.84407	1.18474	.87441	1.14363	.90569	1.10414	.93797	1.06613	.97133	1.02952	50
11	.84457	1.18404	.87492	1.14296	.90621	1.10349	.93852	1.06551	.97189	1.02892	49
12	.84507	1.18334	.87543	1.14229	.90674	1.10285	.93906	1.06489	.97246	1.02832	48
13	.84556	1.18264	.87595	1.14162	.90727	1.10220	.93961	1.06427	.97302	1.02772	47
14	.84606	1.18194	.87646	1.14095	.90781	1.10156	.94016	1.06365	.97359	1.02713	46
15	.84656	1.18125	.87698	1.14028	.90834	1.10091	.94071	1.06303	.97416	1.02653	45
16	.84706	1.18055	.87749	1.13961	.90887	1.10027	.94125	1.06241	.97472	1.02593	44
17	.84756	1.17986	.87801	1.13894	.90940	1.09963	.94180	1.06179	.97529	1.02533	43
18	.84806	1.17916	.87852	1.13828	.90993	1.09899	.94235	1.06117	.97586	1.02474	42
19	.84856	1.17846	.87904	1.13761	.91046	1.09834	.94290	1.06056	.97643	1.02414	41
20	.84906	1.17777	.87955	1.13694	.91099	1.09770	.94345	1.05994	.97700	1.02355	40
21	.84956	1.17708	.88007	1.13627	.91153	1.09706	.94400	1.05932	.97756	1.02295	39
22	.85006	1.17638	.88059	1.13561	.91206	1.09642	.94455	1.05870	.97813	1.02236	38
23	.85057	1.17569	.88110	1.13494	.91259	1.09578	.94510	1.05809	.97870	1.02176	37
24	.85107	1.17500	.88162	1.13428	.91313	1.09514	.94565	1.05747	.97927	1.02117	36
25	.85157	1.17430	.88214	1.13361	.91366	1.09450	.94620	1.05685	.97984	1.02057	35
26	.85207	1.17361	.88265	1.13295	.91419	1.09386	.94676	1.05624	.98041	1.01998	34
27	.85257	1.17292	.88317	1.13228	.91473	1.09322	.94731	1.05562	.98098	1.01939	33
28	.85308	1.17223	.88369	1.13162	.91526	1.09258	.94786	1.05501	.98155	1.01879	32
29	.85358	1.17154	.88421	1.13096	.91580	1.09195	.94841	1.05439	.98213	1.01820	31
30	.85408	1.17085	.88473	1.13029	.91633	1.09131	.94896	1.05378	.98270	1.01761	30
31	.85458	1.17016	.88524	1.12963	.91687	1.09067	.94952	1.05317	.98327	1.01702	29
32	.85509	1.16947	.88576	1.12897	.91740	1.09003	.95007	1.05255	.98384	1.01642	28
33	.85559	1.16878	.88628	1.12831	.91794	1.08940	.95062	1.05194	.98441	1.01583	27
34	.85609	1.16809	.88680	1.12765	.91847	1.08876	.95118	1.05133	.98499	1.01524	26
35	.85660	1.16741	.88732	1.12699	.91901	1.08813	.95173	1.05072	.98556	1.01465	25
36	.85710	1.16672	.88784	1.12633	.91955	1.08749	.95229	1.05010	.98613	1.01406	24
37	.85761	1.16603	.88836	1.12567	.92008	1.08686	.95284	1.04949	.98671	1.01347	23
38	.85811	1.16535	.88888	1.12501	.92062	1.08622	.95340	1.04888	.98728	1.01288	22
39	.85862	1.16466	.88940	1.12435	.92116	1.08559	.95395	1.04827	.98786	1.01229	21
40	.85912	1.16398	.88992	1.12369	.92170	1.08496	.95451	1.04766	.98843	1.01170	20
41	.85963	1.16329	.89045	1.12303	.92224	1.08432	.95506	1.04705	.98901	1.01112	19
42	.86014	1.16261	.89097	1.12238	.92277	1.08369	.95562	1.04644	.98958	1.01053	18
43	.86064	1.16192	.89149	1.12172	.92331	1.08306	.95618	1.04583	.99016	1.00994	17
44	.86115	1.16124	.89201	1.12106	.92385	1.08243	.95673	1.04522	.99073	1.00935	16
45	.86166	1.16056	.89253	1.12041	.92439	1.08179	.95729	1.04461	.99131	1.00876	15
46	.86216	1.15987	.89306	1.11975	.92493	1.08116	.95785	1.04401	.99189	1.00818	14
47	.86267	1.15919	.89358	1.11909	.92547	1.08053	.95841	1.04340	.99247	1.00759	13
48	.86318	1.15851	.89410	1.11844	.92601	1.07990	.95897	1.04279	.99304	1.00701	12
49	.86368	1.15783	.89463	1.11778	.92655	1.07927	.95952	1.04218	.99362	1.00642	11
50	.86419	1.15715	.89515	1.11713	.92709	1.07864	.96008	1.04158	.99420	1.00583	10
51	.86470	1.15647	.89567	1.11648	.92763	1.07801	.96064	1.04097	.99478	1.00525	9
52	.86521	1.15579	.89620	1.11582	.92817	1.07738	.96120	1.04036	.99536	1.00467	8
53	.86572	1.15511	.89672	1.11517	.92872	1.07676	.96176	1.03976	.99594	1.00408	7
54	.86623	1.15443	.89725	1.11452	.92926	1.07613	.96232	1.03915	.99652	1.00350	6
55	.86674	1.15375	.89777	1.11387	.92980	1.07550	.96288	1.03855	.99710	1.00291	5
56	.86725	1.15308	.89830	1.11321	.93034	1.07487	.96344	1.03794	.99768	1.00233	4
57	.86776	1.15240	.89883	1.11256	.93088	1.07425	.96400	1.03734	.99826	1.00175	3
58	.86827	1.15172	.89935	1.11191	.93143	1.07362	.96457	1.03674	.99884	1.00116	2
59	.86878	1.15104	.89988	1.11126	.93197	1.07299	.96513	1.03613	.99942	1.00058	1
60	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	1.00000	1.00000	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	49°		48°		47°		46°		45°		

PHYSICS.

MATTER AND ITS PROPERTIES.

DEFINITIONS.

1. Natural and Physical Science.—Science, which may be defined as a classified knowledge of nature, is divided into *natural* and *physical* science.

Natural science concerns itself with the external form and internal structure of bodies.

Physical science considers only the matter of which these bodies are composed.

ILLUSTRATION.—Geology, mineralogy, botany, and zoology, which investigate the form and structure of the earth, of minerals, of plants, and of animals, respectively, are *natural sciences.*

Physics and chemistry, which consider the properties of matter itself, whether it is light or heavy, hard or soft, combustible or non-combustible, are *physical sciences.*

2. Matter is anything that possesses weight; that is, is acted on by gravitation. In studying matter, physical science considers:

1. The division of which matter is capable.
2. The attractions by which these particles are held together.
3. The motions these particles may have.

3. Division of Matter.—Science recognizes three divisions of matter: *masses*, *molecules*, and *atoms.*

A **mass**, or body, of matter is any portion of matter appreciable by the senses.

A **molecule** is the smallest particle of matter that a body can be divided into without losing its identity.

An **atom** is an indivisible portion of matter.

Atoms unite to form molecules; a collection of molecules forms a mass, or a body.

It has been calculated that the diameter of a molecule is larger than $\frac{1}{12500000000}$ of an inch, and smaller than $\frac{1}{5000000000}$ of an inch.

ILLUSTRATION.—The sun and the grain of sand are both *masses* of matter; the smallest particles of sugar or of salt that still show the properties of these substances, respectively, are *molecules* of sugar or of salt. The still more minute particles of carbon, of hydrogen, and of oxygen, which make up the molecule of sugar, or those of chlorine and of sodium, which compose the molecule of salt, are *atoms*.

4. Attraction of Matter.—Three forms of attraction are recognized in science:

1. That form of attraction called *gravitation*, which is exerted between masses of matter.

2. That form which binds molecules together; called *cohesion*, when the molecules are alike, and *adhesion*, when the molecules are unlike.

3. That form of attraction called *chemical attraction*, or *affinity*, which draws atoms together.

ILLUSTRATION.—The planets are held to the sun, and stones are held to the earth by the attraction of gravitation. The molecules of silver or salt, being alike, are held together by cohesion; those of granite or gunpowder, being unlike, are held together by adhesive attraction. The atoms that form a molecule of sugar or of salt are united by chemical attraction.

Molecules united by cohesion, therefore, form homogeneous matter, while unlike molecules united by adhesion form heterogeneous matter. It is evident that there must be as many kinds of molecules as there are kinds of homogeneous matter.

5. Motions of Matter.—The forms of motion admitted in science are three:

1. **Mass motion,** or visible mechanical motion.

2. **Molecular motion,** the motion of the molecule within

the mass, called *heat*, *light*, *electricity*, or *magnetism*, according to its character.

3. **Atomic motion,** which, though probable, is not yet fully established.

6. Province of Physics.—Physics is that department of physical science which studies the results that flow from the molecular conditions of matter.

ILLUSTRATION.—Weight, which is a consequence of mass attraction; impact, which is a result of mass motion; tenacity, hardness, and elasticity, which depend on cohesion; solution, capillarity, and diffusion, which result from adhesion; heat, light, electricity, and magnetism, which are molecular motions—these are all objects of physical study and investigation.

7. Bodies are, as we have seen, collections of molecules. They exist in three forms or conditions: *solid*, *liquid*, and *gaseous*.

A **solid body** is one whose molecules change their relative positions with great difficulty; as, iron, wood, stone, etc.

A **liquid body** is one whose molecules tend to change their relative position easily. Liquids readily adapt themselves to the vessel that contains them, and their upper surface always tends to become level. Water, mercury, etc. are liquids.

A **gaseous body,** or a gas, is one whose molecules tend to separate from one another; as, air, oxygen, etc.

Gaseous bodies are sometimes called *aeriform* (air-like) bodies. They are divided into two classes: *permanent gases* and *vapors*.

A **permanent gas** is one that remains a gas at ordinary temperatures and pressures.

A **vapor** is a body that at ordinary temperatures is a liquid or solid, but when heat is applied becomes a gas.

One body may, however, exist in all three states; as, for example, mercury, which at ordinary temperatures is a liquid, becomes a solid (freezes) at $-40°$ C., and a vapor at 350° C. Nearly all gases can be liquefied by great cold, and some can even be solidified. By means of heat nearly all solids may be either liquefied or vaporized.

8. Properties of Matter.—All matter possesses certain qualities that are, in scientific language, called *properties*. Some properties are common to all kinds of matter; other properties are possessed only by certain kinds of matter. The former, termed *general properties*, are: *extension*, *impenetrability*, *weight*, *inertia*, *mobility*, *divisibility*, *porosity*, *compressibility*, *expansibility*, *elasticity*, and *indestructibility*. Among the latter, which are known as *special properties*, are the following: *hardness*, *tenacity*, *brittleness*, *malleability*, and *ductility*.

9. Extension is the property of occupying space. Since all bodies must occupy space, it follows that extension is a general property.

10. Impenetrability is the property of matter that prevents two bodies from occupying the same portion of space at the same time.

11. Weight is the measure of the earth's attraction upon a body. All bodies have weight. In former times it was supposed that gases had no weight, since, if unconfined, they tend to move away from the earth; but, nevertheless, they will finally reach a point beyond which they cannot go, being held in suspension by the earth's attraction. Weight is measured by comparison with some standard.

12. Inertia means that a body can neither put itself in motion nor bring itself to rest. To do so, it must be acted on by some force.

13. Mobility means that a body can be changed in position by some force acting on it.

14. Divisibility is that property of matter which indicates that a body may be separated into parts.

15. Porosity is that property of matter which indicates that there is space between the molecules of a body. Molecules of bodies are supposed to be spherical, and hence there is space between them, as there would be between peaches in a basket. It is said that the molecules of water are larger

than those of salt; so that when salt is dissolved in water, its molecules wedge themselves between the molecules of the water, and unless too much salt is added, the water will occupy no more space than it did before. This does not prove that water is penetrable, for the molecules of salt occupy the space between the molecules of water.

Water has been forced through iron by pressure, thus proving that iron is porous.

16. Compressibility is that property of matter which indicates that the molecules of a body may be crowded nearer together, so as to occupy a smaller space.

17. Expansibility is that property of matter which indicates that the molecules of a body may be forced apart, so as to occupy a greater space.

18. Elasticity is that property of matter which indicates that if a body be distorted within certain limits, it will resume its original form when the distorting force is removed. Glass, ivory, and steel are more or less elastic.

19. Indestructibility indicates that matter can never be destroyed. A body may undergo thousands of changes; it may be resolved into its molecules, and its molecules may be resolved into atoms; these atoms may unite with other atoms to form other molecules and bodies, which may be entirely different from the original body; but through all these changes the number of atoms remains the same. The whole number of atoms in the universe is exactly the same now as it was millions of years ago, and will always remain the same. *Matter is indestructible.*

20. Hardness is that property of matter which indicates that some bodies may scratch other bodies. Fluids and gases do not possess hardness. The diamond is the hardest of all substances.

21. Tenacity is that property of matter which indicates that some bodies resist a force tending to pull them apart. Steel is very tenacious.

22. **Brittleness** is that property of matter which indicates that some bodies are easily broken; as, glass, crockery, etc.

23. **Malleability** is that property of matter which indicates that some bodies may be hammered or rolled into sheets. Gold is the most malleable of all substances.

24. **Ductility** is that property of matter which indicates that some bodies may be drawn into wire. Platinum is the most ductile of all substances.

GRAVITATION.

25. Every body in the universe exerts a certain attractive force on every other body, which tends to draw the two bodies together. This attractive force is called *gravitation.*

If a body is held in the hand, a downward pull is felt, and if released, it will fall to the ground. This pull is commonly called *weight*, but it really is the attraction between the earth and the body.

26. **Force of gravity** is a term used to denote the attraction between the earth and bodies upon or near its surface. It always acts in a straight line between the center of the body and the center of the earth.

The force of gravity varies at points on the earth's surface. It is slightly less on the top of a high mountain than at the level of the sea. For this reason, the weight of a body also varies. But if the weight of a body at any place be divided by the force of gravity at that place, the result is called the *mass* of the body.

27. The **mass** of a body is the measure of the actual amount of matter that it contains, and is *always the same.*

Let m = mass of the body;
W = weight of the body;
g = force of gravity at the place where the body was weighed.

Then, $$\text{mass} = \frac{\text{weight of body}}{\text{force of gravity}}, \text{ or } m = \frac{W}{g}. \qquad \textbf{(1.)}$$

28. Law of Gravitation.—*The force of attraction by which one body tends to draw another body towards it is directly proportional to its mass, and inversely proportional to the square of the distance between the centers of the two bodies.*

29. Laws of Weight.—*Bodies weigh most at the surface of the earth. Below the surface, the weight decreases as the distance to the center decreases.*

Above the surface the weight decreases as the square of the distance increases.

ILLUSTRATION.—If the earth's radius is 4,000 miles, a body that weighs 100 pounds at the surface will weigh nothing at the center, since it is attracted in every direction with equal force. At 1,000 miles from the center it will weigh 25 pounds, since

$$4{,}000 : 1{,}000 = 100 : 25.$$

At 2,000 miles from the center it will weigh 50 pounds, since

$$4{,}000 : 2{,}000 = 100 : 50.$$

At 3,000 miles from the center it will weigh 75 pounds, and at the surface, or 4,000 miles from the center, it will weigh 100 pounds. If carried still higher, say 1,000 miles from the surface, or 5,000 miles from the center of the earth, it will weigh 64 pounds, since

$$5{,}000^2 : 4{,}000^2 = 100 : 64.$$

At 4,000 miles from the surface it will weigh 25 pounds, since

$$8{,}000^2 : 4{,}000^2 = 100 : 25.$$

30. Formulas for Gravity Problems.

Let W = weight of a body at the surface;

w = weight of a body at a given distance above or below the surface;

d = distance between the center of the earth and the center of the body;

R = radius of the earth = 4,000 miles.

Formula for weight when the body is below the surface,

$$w R = d W. \qquad (2.)$$

Formula for weight when the body is above the surface,

$$w d^2 = W R^2. \qquad (3.)$$

EXAMPLE 1.—How far below the surface of the earth will a 25-pound ball weigh 9 pounds ?

SOLUTION.—Use formula **2**, $w R = d W$.

Substituting the values of R, W, and w, we have

$$9 \times 4{,}000 = d \times 25,$$

or $$d = \frac{9 \times 4{,}000}{25} = 1{,}440 \text{ miles from the center.}$$

$4{,}000 - 1{,}440 = 2{,}560$ miles below the surface. Ans.

EXAMPLE 2.—If a body weighs 700 pounds at the surface of the earth, at what distance above the earth's surface will it weigh 112 pounds?

SOLUTION.—Use formula **3**, $w d^2 = W R^2$.

Substituting the values of R, W, and w, we have

$$112 \times d^2 = 700 \times 4{,}000^2,$$

or $$d = \sqrt{\frac{700 \times 4{,}000^2}{112}} \text{ or } 10{,}000 \text{ miles.}$$

Therefore, $10{,}000 - 4{,}000 = 6{,}000$ miles above the earth's surface. Ans.

EXAMPLE 3.—The top of Mt. Hercules is said to be 32,000 feet, say 6 miles, above the level of the sea. If a body weighs 1,000 pounds at sea level, what would it weigh if carried to the top of the mountain?

SOLUTION.— $w d^2 = W R^2$, or $w \times 4{,}006^2 = 1{,}000 \times 4{,}000^2$;

whence, $$w = \frac{4{,}000^2 \times 1{,}000}{4{,}006^2} = 997 \text{ pounds.}$$ Ans.

EXAMPLES FOR PRACTICE.

31. Solve the following:

1. How much would 1,000 tons of coal weigh 1 mile below the surface? Ans. 1,999,500 lb.

2. How much would the coal in example 1 weigh 1 mile above the surface? Ans. 1,999,000 lb., nearly.

3. How far above the earth's surface would it be necessary to carry a body in order that it may weigh only half as much? Ans. 1,656.854 mi., nearly.

4. A man weighs 160 pounds at the surface; how much will he weigh 50 miles below the surface? Ans. 158 lb.

5. If a body weighs 100 pounds 400 miles above the earth's surface, how much will it weigh at the surface? Ans. 121 lb.

NOTE.—Use 4,000 miles as the radius of the earth.

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32. *The* **specific gravity** *of a body is the ratio between its weight and the weight of a like volume of water.*

Since gases are so much lighter than water, it is usual to take the specific gravity of a gas as the ratio between the weight of a certain volume of the gas and the weight of the same volume of air.

EXAMPLE.—A cubic foot of cast iron weighs 450 pounds; what is its specific gravity, a cubic foot of water weighing 62.42 pounds?

SOLUTION.—According to the definition, the specific gravity is

$$\frac{450}{62.42} = 7.21. \quad \text{Ans.}$$

33. The specific gravities of different bodies are given in printed tables; hence, if it is desired to know the weight of a body that cannot be conveniently weighed, *calculate its cubical contents and multiply the specific gravity of the body by the weight of a like volume of water, remembering that a cubic foot of water weighs 62.42 pounds.*

EXAMPLE.—How much will 3,214 cubic inches of cast iron weigh? Take its specific gravity as 7.21.

SOLUTION.—Since 1 cubic foot of water weighs 62.42 pounds, 3,214 cubic inches weigh

$$\frac{3,214}{1,728} \times 62.42 = 116.098 \text{ pounds.}$$

Then, $116.098 \times 7.21 = 837.067$ pounds. Ans.

EXAMPLES FOR PRACTICE.

34. Solve the following:

1. If a cubic foot of a certain substance weighs 162½ pounds, what is its specific gravity? Ans. 2.6.

2. A mixture of lead and tin has a specific gravity of 9.28; what is the weight of a cubic inch? Ans. 5.36 oz.

3. What is the weight of a cubic foot of anthracite coal having a specific gravity of 1.55? Ans. 96.751 lb.

4. Find the weight of 100 cubic yards of earth having a specific gravity of 1.75. Ans. 147.467 T.

35. The following tables give the specific gravities of a variety of substances likely to be met with in ordinary practice. The weights per cubic foot are calculated on a basis of 62.5 pounds of water per cubic foot.

TABLE 1.

APPROXIMATE SPECIFIC GRAVITIES OF VARIOUS SUBSTANCES.

METALS.

Substance.	Specific Gravity.	Weight per Cubic Foot in Pounds.
Osmium	22.48	1,405.0
Platinum	21.50	1,343.8
Gold	19.50	1,218.8
Mercury	13.60	850.0
Lead (cast)	11.35	709.4
Silver	10.50	656.3
Copper (cast)	8.79	549.4
Brass	8.38	523.8
Wrought iron	7.68	480.0
Cast iron	7.21	450.0
Steel	7.84	490.0
Tin (cast)	7.29	455.6
Zinc (cast)	6.86	428.8
Antimony	6.71	419.4
Aluminum	2.60	162.5

LIQUIDS.

Substance.	Specific Gravity.	Weight per Cubic Foot in Pounds.
Acetic acid	1.062	66.4
Nitric acid	1.420	88.8
Sulphuric acid	1.841	115.1
Muriatic acid	1.200	75.0
Alcohol	.800	50.0
Turpentine	.870	54.4
Sea-water (ordinary)	1.026	64.1
Milk	1.032	64.5

WOODS.

Substance.	Specific Gravity.	Weight per Cubic Foot in Pounds.
Ash	.845	52.80
Beech	.852	53.25
Cedar	.561	35.06
Cork	.240	15.00
Ebony (American)	1.331	83.19
Lignum vitæ	1.333	83.30
Maple	.750	46.88
Oak (old)	1.170	73.10
Spruce	.500	31.25
Pine (yellow)	.660	41.20
Pine (white)	.554	34.60
Walnut	.671	41.90

GASES.

At 32° F., and Under a Pressure of One Atmosphere.

Substance.	Specific Gravity.	Weight per Cubic Foot in Pounds.
Atmospheric air	1.0000	.08073
Carbonic acid	1.5290	.12344
Carbonic oxide	.9674	.07810
Chlorine	2.4400	.19700
Oxygen	1.1056	.08925
Nitrogen	.9713	.07841
Smoke (bituminous coal)	.1020	.00815
Smoke (wood)	.0900	.00727
*Steam at 212° F.	.4700	.03790
Hydrogen	.0692	.00559

*The specific gravity of steam at any temperature and pressure, compared with air at the same temperature and pressure, is 0.622.

MISCELLANEOUS.

Substance.	Specific Gravity.	Weight per Cubic Foot in Pounds.
Emery	4.00	250
Glass (average)	2.80	175
Chalk	2.78	174
Granite	2.65	166
Marble	2.70	169
Stone (common)	2.52	158
Salt (common)	2.13	133
Soil (common)	1.98	124
Clay	1.93	121
Brick	1.90	118
Plaster of Paris (average)	2.00	125
Sand	1.80	113

36. Determining Specific Gravity of Liquids.—To find the specific gravity of any particular liquid, compared with that of water, it is only necessary to weigh equal bulks and divide the weight of the liquid by the weight of the water. To weigh equal bulks of liquid the simplest and most accurate way is to weigh them in succession in the same vessel, taking care that it is equally full on both occasions.

FIG. 1.

A convenient vessel, commonly used when small quantities are to be weighed, is shown in Fig. 1. It is easily made by blowing a bulb on a glass tube. On that portion of the tube which is narrowed by drawing it out over a flame, a scratch is made with a diamond or a file. The bulb is filled up to the scratch with the liquid, and is then weighed, emptied, cleaned, dried, filled with water, and again weighed. In these experiments care should always be taken that both liquids have the same temperature. This is easily accomplished by

immersing the bulb filled with liquid for some time in water, part of which is later used in the second weighing.

Another form of apparatus for determining the specific gravity of liquids, devised by Dr. Sprengel, consists of an elongated **U** tube, Fig. 2, the ends of which terminate in the two capillary tubes *A*, *B*, bent at right angles in opposite directions. The shorter one *A* is narrower at the end than the longer one. The horizontal part of the wider tube is marked near the bend with a fine line. The **U** tube is filled by suction, the little bulb apparatus, Fig. 3, having been previously attached to the narrow capillary tube by a piece of rubber tubing. It is then detached from the bulb, placed

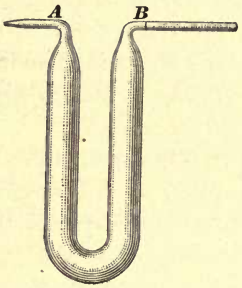

FIG. 2.

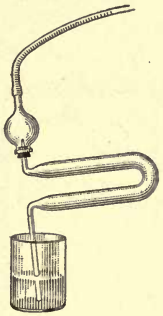
FIG. 3.

in water almost to the bends of the capillary tubes, left there until it has assumed the temperature of the water, and after careful adjustment of the volume of the liquid up to the mark in the wider capillary tube, it is taken out, dried, and weighed.

37. Determining the Specific Gravity of Solids.— The determination of the specific gravity of a solid body is also made according to the principles explained, and may be made with the specific-gravity bottle. The bottle is first weighed full of water; the solid is then placed in the same pan of the balance and its weight determined; finally, the solid is put into the bottle, displacing an equal bulk of water, the weight of which is determined by the loss on again weighing. In this way, the weights of the solid and that of an

equal bulk of water are obtained. The former divided by the latter gives the specific gravity.

EXAMPLE.—A piece of lead weighs..... 73.68 grains.
Glass bottle filled with water weighs..... 297.05 grains.
Total..... 370.73 grains.
After an equal volume of water was displaced by the lead, the weight was..... 364.15 grains.
Hence, the displaced water weighed 6.58 grains.

Therefore, the specific gravity of lead is $\frac{73.68}{6.58} = 11.2-$. Ans.

38. Another ingenious, but not quite as exact, method of finding the specific gravity of solids not easily soluble in water is based on the principle of Archimedes:

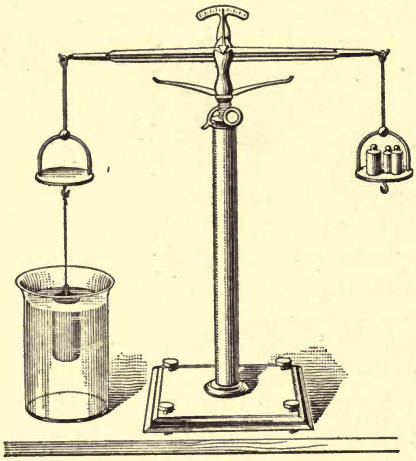

FIG. 4.

When a body is immersed in a fluid, it loses in weight an amount equal to the weight of the fluid it displaces.

This principle is applied as follows: Weigh the body first in the air, then weigh the body in water, suspending it by a string attached to a scale pan, as shown in Fig. 4. *The difference between the two weights will be the weight of an equal volume of water. The ratio of the weight in air to the difference thus found will be the specific gravity.*

Let Sp. Gr. = specific gravity;
W = weight of the solid in air;
W' = weight of the solid in water.

Then the weight of a volume of water equal to the volume of the solid is $W - W'$, and

$$\text{Sp. Gr.} = \frac{W}{W - W'}. \quad (4.)$$

EXAMPLE.—A body in air weighs 36.25 grains, and in water, 30 grains; what is its specific gravity?

SOLUTION.—Using formula **4**, we obtain

$$\text{Sp. Gr.} = \frac{W}{W-W'} = \frac{36.25}{36.25-30} = \frac{36.25}{6.25} = 5.8. \quad \text{Ans.}$$

39. If the body is lighter than water, a piece of iron or other heavy substance must be attached to it, sufficiently heavy to sink both. *Then weigh both bodies in air and both in water. Weigh both separately in air, and weigh the heavier body in water. Subtract the weight of the bodies in water from their weight in air, and the result will be the weight of a volume of the water equal to the volume of the two bodies. Find the difference of the weights of the heavy body in air and in water, and the result will be the weight of a volume of water equal to the volume of the heavy body. Subtract this last result from the former, and the result will be the weight of a volume of water equal to the volume of the light body. The weight of the light body in air divided by the weight of its equal volume of water is the specific gravity of the light body.*

Let W = weight of both bodies in air;
W' = weight of both bodies in water;
w = weight of light body in air;
W_1 = weight of heavy body in air;
W_2 = weight of heavy body in water.

Then, the specific gravity of the light body is given by

$$\text{Sp. Gr.} = \frac{w}{(W-W')-(W_1-W_2)}. \qquad (5.)$$

EXAMPLE.—A piece of cork weighs 4.8 grains in air. A piece of cast iron weighs 36 grains in air and 31 grains in water. The weight of the iron and cork together in water is 15 grains; what is the specific gravity of the cork? of the cast iron?

SOLUTION.—Substituting in formula **5** the values given,

$$\text{Sp. Gr.} = \frac{w}{(W-W')-(W_1-W_2)} = \frac{4.8}{(40.8-15)-(36-31)}$$

$$= \frac{4.8}{20.8} = 0.23, \text{ the specific gravity of the cork.} \quad \text{Ans.}$$

By formula **4**,

$$\text{Sp. Gr.} = \frac{W}{W-W'} = \frac{36}{36-31} = 7.2, \text{ specific gravity of the iron.} \quad \text{Ans.}$$

40. Determining Specific Gravity of Gases.—The method of finding the specific gravity of a gas is of great importance to the chemist. The theory of the operation is as simple as if liquids are concerned, but as the process is more delicate and involves certain corrections for differences of temperature and pressure, it appears preferable to defer the consideration of this matter for the present.

41. Hydrometers.—Instruments called *hydrometers* are in general use for determining quickly and accurately the specific gravities of liquids and some forms of solids. There are two kinds; viz.,

1. Hydrometers of constant weight, as **Beaumé's.**
2. Hydrometers of constant volume, as **Nicholson's.**

42. A hydrometer of constant weight is shown in Fig. 5. It consists of a glass tube, near the bottom of which are two bulbs. The lower and smaller bulb is loaded with mercury or shot, so as to cause the instrument to remain in a vertical position when placed in the liquid. The upper bulb is filled with air, and its volume is such that the whole instrument is lighter than an equal volume of water.

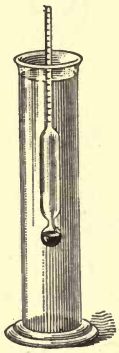

FIG. 5.

The point to which the hydrometer sinks when placed in water is usually marked, the tube being graduated above and below in such a manner that the specific gravity of the liquid can be read directly. It is customary to have two instruments: one with the zero point near the top of the stem, for use in liquids heavier than water, and the other with the zero point near the bulb, for use in liquids lighter than water.

These instruments are more commonly used for determining the degree of concentration or dilution of certain liquids, as acids, alcohol, milk, solutions of sugar, etc., rather than their actual specific gravities. They are then known as *acidimeters*, *alcoholometers*, *lactometers*, *saccharimeters*, etc., according to the use to which they are put.

43. Nicholson's Hydrometer.—This instrument is shown in Fig. 6. It consists of a hollow cylinder carrying at its lower end a basket d heavy enough to keep the apparatus upright when placed in water. At the top of the cylinder is a vertical rod, to which is attached a shallow pan a for holding weights, etc. The cylinder must be of such size that the apparatus may be so much lighter than water that a certain weight W must be placed in the pan to sink it to a given point c on the rod. The body whose specific gravity it is desired to find must weigh less than W. It is placed in the pan a and enough weight w is added to sink the point c to the water level. It is evident that the weight of the given body is $W-w$. The body is now removed from the pan a and placed in the basket d, an additional weight being added to sink the point c to the water level. Represent the weight now in the pan by W'. The difference $W'-w$ is the weight of a volume of water equal to the volume of the body. Hence,

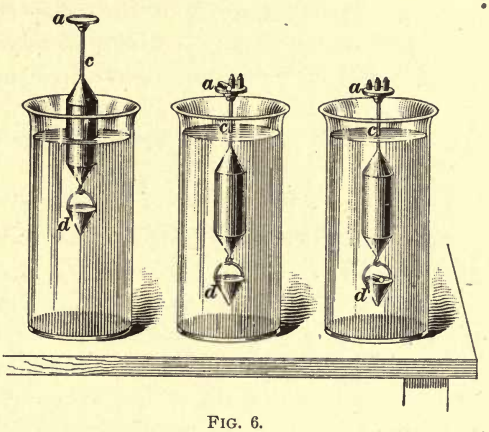

FIG. 6.

$$\text{Sp. Gr.} = \frac{W-w}{W'-w}. \qquad \textbf{(6.)}$$

EXAMPLE.—The weight necessary to sink the hydrometer to the point c is 16 ounces; the weight necessary when the body is in the pan a is 7.3 ounces, and when the body is in the basket d, 10 ounces; what is the specific gravity of the body?

SOLUTION.—Sp. Gr. $= \frac{W-w}{W'-w} = \frac{16-7.3}{10-7.3} = \frac{8.7}{2.7} = 3.222$. Ans.

44. Archimedes' principle gives a very easy and accurate method of finding the volume of an irregularly shaped body. Thus, subtract its weight in water from its weight in air;

divide by .03617 and the result will be the volume in cubic inches; or divide by 62.5 and the result will be the volume in cubic feet.

If the specific gravity of the body is known, its cubical contents can be found by dividing its weight by its specific gravity, and then dividing again by either .03617 or 62.5.

EXAMPLE.—A certain body has a specific gravity of 4.38 and weighs 76 pounds. How many cubic inches are there in the body?

SOLUTION.— $\frac{76}{4.38 \times .03617} = 479.72$ cu. in. Ans.

45. Since the weight of a cubic foot of water varies for different temperatures, and with the amount of impurities it contains, it is necessary to have some standard when getting the specific gravity. This standard is pure distilled water at its maximum density, which occurs at a temperature of 39.2° F. At this temperature water weighs 62.425 pounds per cubic foot; but for ordinary calculations it is customary to take it as weighing 1,000 ounces, or 62.5 pounds per cubic foot.

PROPERTIES OF AIR AND GASES.

TENSION OF GASES.

46. The most striking feature concerning gases is that, *no matter how small the quantity may be, they will always fill the vessels that contain them.* If a bladder or football is partly filled with air and placed under a glass jar (called a *receiver*), from which the air has been exhausted, the bladder or football will immediately expand, as shown in Fig. 7. The force that a gas always exerts when confined in a limited space is called *tension.* The word "tension" in this

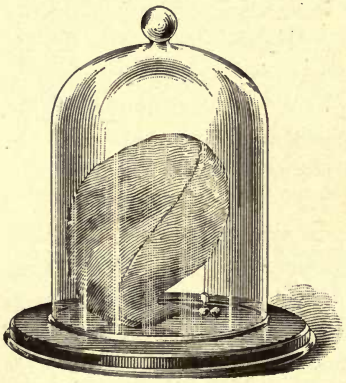

FIG. 7.

case means pressure, and is only used in this sense in reference to gases.

47. As water is the most common liquid, so is air the most common gas. It was supposed by the ancients that air was imponderable; i. e., that it had no weight, and not until the year 1650 was the contrary proved. A cubic inch of air, under ordinary conditions, weighs nearly .31 grain. The ratio of the weight of air to water is about 1 : 774, which means that air is only $\frac{1}{774}$ as heavy as water. It is a well known fact that if a body is immersed in water and weighs less than the volume of water it displaces, this body will rise, extend partly out of the water, and float. The same is true, to a certain extent, of air. If a vessel made of light material is filled with a gas lighter than air, so that the total weight of both, vessel and gas, is less than the weight of the volume of air that they displace, the vessel will rise; it is on this principle that balloons are made.

48. Since the air has weight, it is evident that the enormous quantity of air that constitutes the atmosphere must exert a considerable pressure upon the earth. We can easily notice this atmospheric pressure upon our persons. If we place the hand over the open mouth of the receiver of a powerful air pump, we find that the hand is pressed downwards with great force. The total atmospheric pressure that the human body has to support amounts to several tons, but this pressure is not felt under ordinary circumstances because it is exerted equally in every direction.

49. This pressure of air may easily be proved by taking a long glass tube, closed at one end, and filling it with mercury. If the finger is placed over the open end, so as to keep the mercury from running out, and the tube is inverted and placed in a cup of mercury, as shown in Fig. 8, the mercury will fall, then rise, and after a few oscillations will come to rest at a height above the top of the mercury in the cup equal to about 30 inches. This height will always be constant under similar atmospheric conditions. Now, if the

atmosphere has weight, it must press upon the upper surface of the mercury in the glass with equal intensity upon every square unit, except upon that part of the surface occupied by the tube. According to Pascal's law, *the pressure per unit of area exerted anywhere upon a mass of liquid is transmitted undiminished in all directions, and acts with the same force upon all surfaces in a direction at right angles to those surfaces.* Therefore the pressure of the atmosphere is transmitted in all directions. There being nothing in the tube except mercury to counterbalance the upward pressure of the air, the mercury falls in the tube until it exerts a downward pressure on the upper surface of the mercury in the cup sufficiently great to counterbalance the upward pressure produced by the atmosphere. In order that there shall be equilibrium, the pressure of the air per unit of area on the upper surface of the mercury in the glass must equal the pressure (weight) exerted per unit of area by the mercury inside of the tube. Suppose the area of the inside of the tube is 1 square inch; then, since mercury is 13.6 times as heavy as water, the weight of the mercurial column is $.03617 \times 13.6 \times 30 = 14.7574$ pounds. The actual height of the mercury is a little less than 30 inches, and the actual weight of a cubic inch of distilled water is a little less than .03617 pound. Taking these considerations into account, the average weight of the mercurial column at the level of the sea is 14.69 pounds, or, as it is usually expressed, 14.7 pounds. Since this weight, exerted upon 1 square inch of the liquid in the glass, just produced equilibrium, it is plain

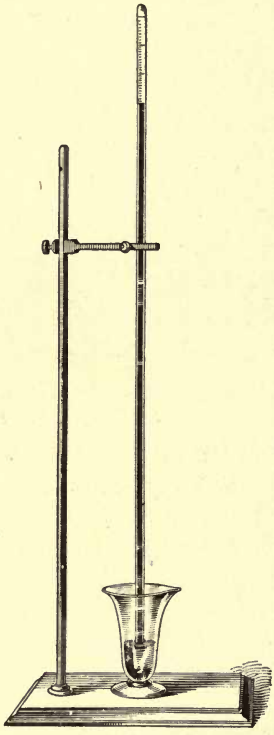

FIG. 8.

that the pressure of the outside air is 14.7 pounds upon every square inch of surface.

50. Vacuum.—The space between the upper end of the tube and the upper surface of the mercury is called a *vacuum*, meaning that it is an entirely empty space, and does not contain any substance—solid, liquid, or gaseous. If it contained a gas of some kind, no matter how small the quantity might be, it would expand, filling the space, and its tension would cause the column of mercury to fall and become shorter, according to the amount of gas present. The space is then called a *partial* vacuum. If the mercury falls 1 inch, so that the column is only 29 inches high, we say, in ordinary language, that there are *29 inches of vacuum*. If it falls 8 inches, we say that there are 22 inches of vacuum; if it falls 16 inches, we say that there are 14 inches of vacuum; etc. Hence, when the vacuum gauge of a condensing engine shows 26 inches of vacuum, there is enough air in the condenser to produce a pressure of

$$\frac{30-26}{30}\times 14.7 = \frac{4}{30}\times 14.7 = 1.96 \text{ lb. per sq. in.}$$

In all cases where the mercurial column is used to measure a vacuum, the height of the column in inches gives the number of inches of vacuum. Thus, if the column were 5 inches high, or the vacuum gauge showed 5 inches, the vacuum would be 5 inches.

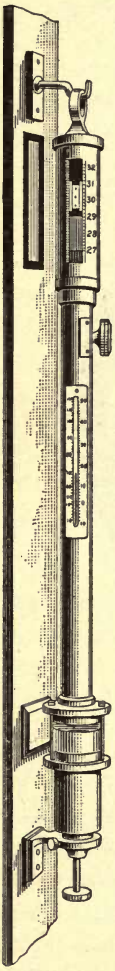

FIG. 9.

If the tube had been filled with water instead of mercury, the height of the column of water to balance the pressure of the atmosphere would have been $30 \times 13.6 = 408$ inches $= 34$ feet. This means that if a tube were filled with water, inverted, and placed in a dish of water in a manner similar to the experiment made with the mercury, the resulting height of the column of water would be 34 feet.

51. The Barometer.—The barometer is an instrument for measuring the pressure of the atmosphere. There are two kinds in general use—the *mercurial* and the *aneroid barometer*. The **mercurial barometer** is shown in Fig. 9. The principle is the same as in the case of the inverted tube shown in Fig. 8. The tube and cup at the bottom are protected by a brass or iron casing. At the top of the tube is a graduated scale that can be read to $\frac{1}{1000}$ of an inch by

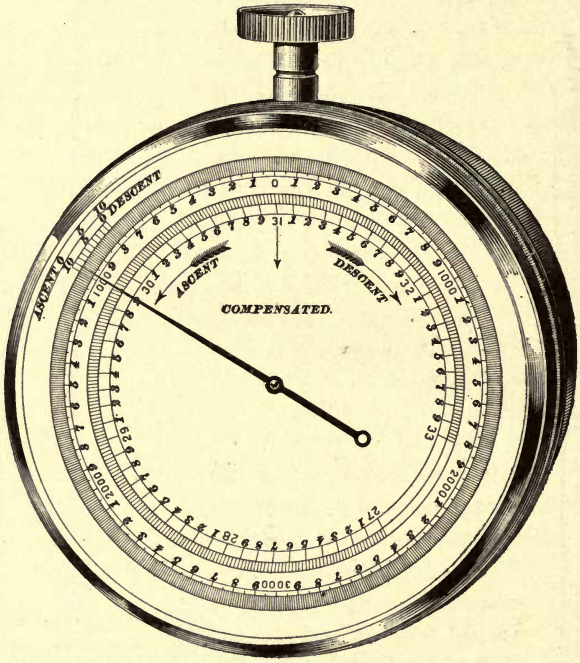

FIG. 10.

means of a vernier. Attached to the casing is an accurate thermometer for determining the temperature of the outside air at the time the barometric observation is taken. This is necessary, since mercury expands when the temperature is increased, and contracts when the temperature falls; for this reason a standard temperature is assumed, and all barometer readings are reduced to this temperature. This standard

temperature is usually taken at 32° F., at which temperature the height of the mercurial column is 30 inches. Another correction is made for the altitude of the place above sea level, and a third correction for the effects of capillary attraction.

52. In Fig. 10 is shown an illustration of an **aneroid barometer.** These instruments are made in various sizes, from the size of a large watch up to an 8 or 10 inch face. The barometer consists of a cylindrical box of metal, with a top of thin, elastic, corrugated metal. The air is removed from the box. When the atmospheric pressure increases, the top is pressed inwards, and when it is diminished, the top is pressed outwards by its own elasticity, aided by a spring beneath. These movements of the cover are transmitted and multiplied by a combination of delicate levers that act upon an index hand and cause it to move either to the right or left over a graduated scale. These barometers are self-correcting (compensated) for variations in temperature. They are very portable, occupying but a small space, and are so delicate that they are said to show a difference in the atmospheric pressure when transferred from the table to the floor. They must be handled with care, as they are easily injured. The mercurial barometer is the standard.

53. With air, as with water, the lower we go, the greater the pressure, and the higher we go, the less the pressure. At the level of the sea, the height of the mercurial column is about 30 inches; at 5,000 feet above the sea, it is 24.7 inches; at 10,000 feet, it is 20.5 inches; at 15,000 feet, it is 16.9 inches; at 3 miles, it is 16.4 inches; and at 6 miles, it is 8.9 inches.

54. The density also varies with the altitude; that is, a cubic foot of air at an elevation of 5,000 feet above the sea level will not weigh as much as a cubic foot at sea level. This is proved conclusively by the fact that at a height of $3\frac{1}{2}$ miles the mercurial column measures but 15 inches, indicating that half the weight of the entire atmosphere is below

that. It is known that the height of the earth's atmosphere is at least 50 miles; hence, the air just before reaching the limit must be in an exceedingly rarefied state. It is by means of barometers that great heights are measured. The aneroid barometer has the heights marked on the dial, so that it can be read directly. With the mercurial barometer the heights must be calculated from the reading.

55. The atmospheric pressure is everywhere present, and presses all objects in all directions with equal force. If a book is laid upon the table, the air presses upon it in every direction with an equal average force of 14.7 pounds per square inch. It would seem as though it would take considerable force to raise a book from the table, since if the size of the book were 8 inches by 5 inches, the pressure upon it is $8 \times 5 \times 14.7 = 588$ pounds; but there is an equal pressure beneath the book to counteract the pressure on the top. It would now seem as though it would require a great force to open the book, since there are two pressures of 588 pounds each, acting in opposite directions, and tending to crush the book; so it would, but for the fact that there is a layer of air between each leaf, acting upwards and downwards with a pressure of 14.7 pounds per square inch.

If two metal plates are made as perfectly smooth and flat as it is possible to get them, and the edge of one be laid upon the edge of the other, so that one may be slid upon the other and the air thus excluded, it will take an immense force, compared with the weight of the plates, to separate them. This is because the full pressure of 14.7 pounds per square inch is then exerted upon each plate, with no counteracting equal pressure between them.

If a piece of flat glass is laid upon a flat surface that has been previously moistened with water, it will require considerable force to separate them; this is because the water helps to fill up the pores in the flat surface and in the glass, and thus creates a partial vacuum between the glass and the surface, thereby reducing the counterpressure beneath the glass.

EXPANSION OF GASES.

56. In Fig. 8 the space above the column of mercury was said to be a vacuum, and that if any gas or air was present, it would expand, its tension forcing the column of mercury downwards. If enough gas is admitted to cause the mercury to stand at 15 inches, the tension of the gas is evidently $\frac{14.7}{2} = 7.35$ pounds per square inch, since the pressure of the outside air of 14.7 pounds per square inch only balances 15 inches, instead of 30 inches of mercury; that is, it balances only half as much as it would if there were no gas in the tube; therefore, the pressure (tension) of the gas in the tube is 7.35 pounds. If more gas is admitted, until the top of the mercurial column is just level with the mercury in the cup, the gas in the tube has then a tension equal to the outside pressure of the atmosphere. Suppose that the bottom of the tube is fitted with a piston, and that the total length of the inside of the tube is 36 inches. If the piston is shoved upwards, so that the space occupied by the gas is 18 inches long, instead of 36 inches, the temperature remaining the same as before, it will be found that the tension of the gas within the tube is 29.4 pounds per square inch. It will be noticed that the volume occupied by the gas is only half that in the tube before the piston was moved, while the pressure is twice as great, since $14.7 \times 2 = 29.4$ pounds. If the piston is pushed farther up, so that the space occupied by the gas is only 9 inches, instead of 18 inches, the temperature still remaining the same, the pressure will be found to be 58.8 pounds per square inch. The volume has again been reduced one-half, and the pressure increased 2 times, since $29.4 \times 2 = 58.8$ pounds. The space now occupied by the gas is 9 inches long, whereas, before the piston was moved, it was 36 inches long; as the tube was assumed to be of uniform diameter throughout its length, the volume is now $\frac{9}{36} = \frac{1}{4}$ of its original volume, and its pressure is $\frac{58.8}{14.7} = 4$ times its original pressure. Moreover, if the temperature of the confined gas remains the same, the pressure

and volume will always vary in a similar way. The law that states these effects is called *Mariotte's law*, and is as follows:

57. Mariotte's Law.—*The temperature remaining the same, the volume of a given quantity of gas varies inversely as the pressure.*

The meaning of this is: If the volume of the gas is diminished to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, etc. of its former volume, the tension will be increased 2, 3, 5, etc. times; or, if the outside pressure is increased 2, 3, 5, etc. times, the volume of the gas will be diminished to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, etc. of its original volume, the temperature remaining constant. It also means that, if a gas is under a certain pressure, and the pressure is diminished to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{10}$, etc. of its original pressure, the volume of the confined gas will be increased 2, 3, 10, etc. times—its tension decreasing at the same rate.

Suppose 3 cubic feet of air to be under a pressure of 60 pounds per square inch in a cylinder fitted with a movable piston; then, the product of the volume and pressure is $3 \times 60 = 180$. Let the volume be increased to 6 cubic feet, then the pressure will be 30 pounds per square inch, and $30 \times 6 = 180$, as before. Let the volume be increased to 24 cubic feet, it is then $\frac{24}{3} = 8$ times its original volume, and the pressure is $\frac{1}{8}$ of its original pressure, or $60 \times \frac{1}{8} = 7\frac{1}{2}$ pounds, and $24 \times 7\frac{1}{2} = 180$, as in the two preceding cases. It will now be noticed that if a gas is enclosed within a confined space, and allowed to expand without losing any heat, *the product of the pressure and the corresponding volume for one position of the piston is the same as for any other position of the piston.* If the piston were to compress the air, the same result would be obtained.

Let p = pressure for one position of the piston;
p_1 = pressure for any other position of the piston;
v = volume corresponding to the pressure p;
v_1 = volume corresponding to the pressure p_1.

Then,
$$p\,v = p_1\,v_1. \qquad (7.)$$

Knowing the volume and the pressure for any position of the piston and the volume for any other position, the pressure

may be calculated, or if the pressure is known for any other position, the volume may be calculated.

EXAMPLE 1.—If 1.875 cubic feet of air be under a pressure of 72 pounds per square inch, what will be the pressure when the volume is increased (*a*) to 2 cubic feet? (*b*) to 3 cubic feet? (*c*) to 9 cubic feet?

SOLUTION.—Solving formula **7** for p_1, the unknown pressure,

(*a*) $$p_1 = \frac{p\,v}{v_1} = \frac{72 \times 1.875}{2} = 67\tfrac{1}{2} \text{ lb. per sq. in.} \quad \text{Ans.}$$

(*b*) $$p_1 = \frac{72 \times 1.875}{3} = 45 \text{ lb. per sq. in.} \quad \text{Ans.}$$

(*c*) $$p_1 = \frac{72 \times 1.875}{9} = 15 \text{ lb. per sq. in.} \quad \text{Ans.}$$

EXAMPLE 2.— 10 cubic feet of air have a tension of 5.6 pounds per square inch; what is the volume when the tension is (*a*) 4 pounds? (*b*) 8 pounds? (*c*) 25 pounds? (*d*) 100 pounds?

SOLUTION.—Solving formula **7** for v_1,

(*a*) $$v_1 = \frac{p\,v}{p_1} = \frac{5.6 \times 10}{4} = 14 \text{ cu. ft.} \quad \text{Ans.}$$

(*b*) $$v_1 = \frac{5.6 \times 10}{8} = 7 \text{ cu. ft.} \quad \text{Ans.}$$

(*c*) $$v_1 = \frac{5.6 \times 10}{25} = 2.24 \text{ cu. ft.} \quad \text{Ans.}$$

(*d*) $$v_1 = \frac{5.6 \times 10}{100} = .56 \text{ cu. ft.} \quad \text{Ans.}$$

58. There are two ways of measuring the pressure of a gas: by means of an instrument called a *manometer*, and by means of a *gauge*. The manometer generally used is practically the same as a mercurial barometer, except that the tube is much longer, so that pressures equal to several atmospheres may be measured, and is enlarged and bent into a **U** shape at the lower end; both lower and upper ends are open, the lower end being connected to the vessel containing the gas, whose pressure it is desired to measure. The gauge is so common that no description of it will be given here. With both the manometer described above and the gauge, the pressures recorded are the amounts by which they exceed the atmospheric pressure, and are called the *gauge pressures*. To find the real pressure, called the *absolute pressure*, the

atmospheric pressure must be added to the gauge pressure. In all formulas in which the pressure of a gas or steam is used, the absolute pressure must be used, unless the gauge pressure is distinctly specified as being the proper pressure to use. For convenience, all pressures given in this Instruction Paper and in the accompanying Question Paper will be absolute pressures, and the word *absolute* will be omitted to avoid its constant repetition.

59. As a necessary consequence of Mariotte's law, it may be stated that *the density of a gas varies directly as the pressure, and inversely as the volume; that is, the density increases as the pressure increases, and decreases as the volume increases*

This is evident, since, if a gas has a tension of 2 atmospheres, or $14.7 \times 2 = 29.4$ pounds per square inch, it will weigh twice as much as the same volume would if the tension were 1 atmosphere, or 14.7 pounds per square inch. For, let the volume be increased until it is twice as great as the original volume, the tension will then be 1 atmosphere. The total weight of the gas has not been changed, but there are now 2 cubic feet for every cubic foot of the original volume, and the weight of 1 cubic foot now is only half as great as before. Thus, the density decreases as the volume increases, and as an increase of pressure causes a decrease of volume, the density increases as the pressure increases.

Let D = density corresponding to the pressure p and volume v;

D_1 = density corresponding to the pressure p_1 and volume v_1.

Then, $p : D = p_1 : D_1$, or $p D_1 = p_1 D$, **(8.)**

and $v : D_1 = v_1 : D$, or $v D = v_1 D_1$. **(9.)**

60. Since the weight is proportional to the density, the weights may be used in place of the densities in formulas **8** and **9.**

Let W = weight of a cubic foot of air or other gas, whose volume is v and pressure is p;

W_1 = weight of a cubic foot when the volume is v_1 and pressure is p_1.

Then,
$$pW_1 = p_1W. \quad \textbf{(10.)}$$
$$vW = v_1W_1. \quad \textbf{(11.)}$$

EXAMPLE 1.—The weight of 1 cubic foot of air at a temperature of 60° F., and under a pressure of 1 atmosphere (14.7 pounds per square inch), is .0763 pound; what will be the weight per cubic foot if the air is compressed until the tension is 5 atmospheres, the temperature still being 60° F.?

SOLUTION.—Applying formula **10**, $pW_1 = p_1W$,

or $1 \times W_1 = 5 \times .0763$.

Hence, $W_1 = .3815$ lb. per cu. ft. Ans.

EXAMPLE 2.—If in the last example the air had expanded until the tension was 5 pounds per square inch, what would have been its weight per cubic foot?

SOLUTION.—Applying formula **10**, $pW_1 = p_1W$. Here, $p = 14.7$, $p_1 = 5$, and $W = .0763$. Hence, $14.7 \times W_1 = 5 \times .0763$,

or
$$W_1 = \frac{.3815}{14.7} = .02595 \text{ lb. per cu. ft.} \quad \text{Ans.}$$

EXAMPLE 3.—If 6.75 cubic feet of air at a temperature of 60° F., and a pressure of 1 atmosphere, are compressed to 2.25 cubic feet (the temperature still remaining 60° F.), what is the weight of a cubic foot of the compressed air?

SOLUTION.—Applying formula **11**,

$$vW = v_1W_1, \text{ or } 6.75 \times .0763 = 2.25 \times W_1.$$

Hence,
$$W_1 = \frac{6.75 \times .0763}{2.25} = .2289 \text{ lb. per cu. ft.} \quad \text{Ans.}$$

61. In all that has been said before, it has been stated that the temperature was constant; the reason for this will now be explained. Suppose 5 cubic feet of air to be confined in a cylinder placed in a vacuum, so that there will be no pressure due to the atmosphere, and suppose the cylinder to be fitted with a piston weighing, say, 100 pounds and having an area of 10 square inches. The tension of the gas will be $\frac{100}{10} = 10$ pounds per square inch. Suppose that the temperature of the air is 32° F., and that it is heated until the

temperature is 33° F., i. e., until the temperature is increased 1°, it will be found that the piston has risen a certain amount, and, consequently, the volume has increased, while the pressure is the same as before, or 10 pounds per square inch. If more heat is applied, until the temperature of the gas is 34° F., it will be found that the piston has again risen, and the volume again increased, while the pressure still remains the same. It will be found that for every increase of temperature there will be a corresponding increase of volume. The law that expresses this change is called *Gay-Lussac's law*, and is as follows:

62. Gay-Lussac's Law.—*If the pressure remains constant, every increase of temperature of 1° F. produces in a given quantity of gas an expansion of $\frac{1}{492}$ of its volume at 32° F.*

If the pressure remains constant, it will also be found that every decrease of temperature of 1° F. will cause a decrease of $\frac{1}{492}$ of the volume at 32° F.

Let v = original volume of gas;
v_1 = final volume of gas;
t = temperature corresponding to volume v;
t_1 = temperature corresponding to volume v_1.

Then, $$v_1 = v\left(\frac{460 + t_1}{460 + t}\right). \qquad (12.)$$

That is, *the volume of gas after heating (or cooling) is equal to the original volume multiplied by 460 plus the final temperature, divided by 460 plus the original temperature.*

EXAMPLE.— 5 cubic feet of air at a temperature of 45° are heated under constant pressure up to 177°; what is its final volume?

SOLUTION.—Applying formula **12**,

$$v_1 = v\left(\frac{460 + t_1}{460 + t}\right) = 5\left(\frac{460 + 177}{460 + 45}\right) = 6.307 \text{ cu. ft.} \quad \text{Ans.}$$

63. Suppose a certain volume of gas to be confined in a vessel so that it cannot expand; in other words, suppose the piston of the cylinder before mentioned to be so fastened that it cannot move. Let a gauge be placed on the cylinder so that the tension of the confined gas can be registered. If

the gas is heated, it will be found that for every increase of temperature of 1° F. there will be a corresponding increase of $\frac{1}{492}$ of the tension. That is, the volume remaining constant, the tension increases $\frac{1}{492}$ of the original tension for every degree rise of temperature.

Let p = original tension;
t = corresponding temperature;
p_1 = final tension;
t_1 = final temperature.

Then, $$p_1 = p\left(\frac{460 + t_1}{460 + t}\right). \qquad \textbf{(13.)}$$

That is, *if a certain quantity of gas be heated (or cooled) from t° to t_1°, the volume remaining constant, the resulting tension p_1 will be equal to the original tension multiplied by 460 plus the final temperature, divided by 460 plus the original temperature.*

EXAMPLE.—If a certain quantity of air is heated under constant volume from 45° to 177°, what is the resulting tension, the original tension being 14.7 pounds per square inch?

SOLUTION.—Applying formula **13**,

$$p_1 = p\left(\frac{460 + t_1}{460 + t}\right) = 14.7\left(\frac{460 + 177}{460 + 45}\right) = 18.542 \text{ lb. per sq. in.} \quad \text{Ans.}$$

64. Absolute Temperature.—According to the modern and now generally accepted theory of heat, the atoms and molecules of all bodies are in an incessant state of vibration. The vibratory movement in the liquids is faster than in the solids, and in the gases, faster than in either of the other two. Any increase of heat increases the vibrations, and a decrease of heat decreases them. From experiments and calculations, it has been concluded that at 460° below zero on the Fahrenheit scale all these vibrations cease. This temperature is called the *absolute zero*, and all temperatures reckoned from this point are called the *absolute temperatures*. The point of absolute zero has never been reached, the lowest recorded temperature being about 393° F. below zero, but, nevertheless, it has a meaning, and is used in many formulas, being nearly always denoted by T. The ordinary temperatures

are denoted by t. When the word temperature alone is used, the meaning is the same as ordinarily used, but when absolute temperature is specified, 460° F. must be added to the ordinary temperature. The absolute temperature corresponding to 212° F. is $460° + 212° = 672°$ F. If the absolute temperature is given, the ordinary temperature may be found by subtracting 460° from the absolute temperature. For example, the absolute temperature being 520° F., the ordinary temperature is $520° - 460° = 60°$.

65. The relation between the pressure, volume, and temperature of air is given by the following formula:

Let p = pressure of air in pounds per square inch;
v = volume of 1 pound of air in cubic feet;
V = volume of W pounds of air in cubic feet;
T = absolute temperature;
W = weight of air in pounds.

Then, $$p\,v = .37052\,T. \qquad \textbf{(14.)}$$

That is, *the pressure in pounds per square inch, multiplied by the volume of a pound of air in cubic feet, is equal to .37052 times the absolute temperature corresponding to the pressure p and volume v.*

In this formula the weight of the air is 1 pound.

EXAMPLE 1.—The pressure upon 9 cubic feet of air weighing 1 pound is 20 pounds per square inch; what is the temperature?

SOLUTION.—Applying formula **14**,

$$p\,v = .37052\,T, \text{ or } 20 \times 9 = .37052\,T.$$

Hence, $$T = \frac{180}{.37052} = 485.8°, \text{ nearly.}$$

$485.8° - 460° = 25.8°$, the temperature. Ans.

EXAMPLE 2.—What is the volume of 1 pound of air whose temperature is 60° F. under a pressure of 1 atmosphere?

SOLUTION.—Applying formula **14**, $p\,v = .37052\,T$, and substituting,

$$14.7 \times v = .37052 \times (460 + 60) = .37052 \times 520,$$

or $$v = \frac{.37052 \times 520}{14.7} = 13.107 \text{ cu. ft.} \quad \text{Ans.}$$

66. If the weight of the air is greater or less than 1 pound, the following formula must be used:

$$pV = .37052\,WT. \qquad \textbf{(15.)}$$

That is, *the pressure in pounds per square inch multiplied by the volume in cubic feet is equal to .37052 times the weight in pounds multiplied by the absolute temperature.*

EXAMPLE 1.— 3 cubic feet of air weighing .35 pound are under a pressure of 48 pounds per square inch; what is the temperature of the air?

SOLUTION.—Applying formula **15**, $pV = .37052\,WT$, and substituting,

$$48 \times 3 = .37052 \times .35 \times T,$$

or
$$T = \frac{48 \times 3}{.37052 \times .35} = 1{,}110.4°.$$

Then, $1{,}110.4° - 460° = 650.4°$. Ans.

EXAMPLE 2.—What is the weight of 1 cubic foot of air at a temperature of 32°, and under a pressure of 1 atmosphere?

SOLUTION.—Applying formula **15**, $pV = .37052\,WT$, and substituting,

$$14.7 \times 1 = .37052 \times (460 + 32) \times W,$$

or
$$W = \frac{14.7}{.37052 \times 492} = .0806382 \text{ lb.} \quad \text{Ans.}$$

If the pressure be taken as 14.69856 pounds per square inch, and the absolute zero as 459.4° instead of 460° below zero, and if .370514 be used instead of .37052 (more exact values), the weight of 1 cubic foot is

$$\frac{14.69856}{.370514 \times 491.4} = .08073 \text{ lb., nearly.}$$

EXAMPLE 3.—What is the exact volume of 1 pound of air at a temperature of 32°, and at a pressure of 1 atmosphere? Take absolute zero as 459.4°, and the pressure as 14.69856 pounds per square inch.

SOLUTION.— $pV = .370514\,WT$,

or
$$14.69856 \times V = .370514 \times 1 \times (459.4 + 32).$$

$$V = \frac{.370514 \times 491.4}{14.69856} = 12.387 \text{ cu. ft.} \quad \text{Ans.}$$

67. If in the formula $pV = .37052\,WT$, both sides of the equation be divided by T (which, of course, does not alter the equality), there results the expression $\frac{pV}{T} = .37052\,W$. Let p_1, V_1, and T_1 represent the pressure, volume, and temperature of the same weight of air in another state; then, $p_1V_1 = .37052\,WT_1$. Dividing both sides by T_1, $\frac{p_1V_1}{T_1} = .37052\,W$. Therefore, since $\frac{pV}{T}$ and $\frac{p_1V_1}{T_1}$ are equal

to the same thing (i. e., $.37052\,W$), they are equal to each other, and

$$\frac{pV}{T} = \frac{p_1 V_1}{T_1}. \qquad \textbf{(16.)}$$

This very important formula is the complete expression of Gay-Lussac's law, and is true for any of the so called permanent gases. It was from this formula that formulas **12** and **13** were derived. Thus, let the pressure be constant; then, $p = p_1$, and $\frac{pV}{T} = \frac{pV_1}{T_1}$, or $V_1 = \frac{VT_1}{T} = V\left(\frac{460+t_1}{460+t}\right)$. Similarly, letting the volume be constant, $V = V_1$, and $\frac{pV}{T} = \frac{p_1 V}{T_1}$, or $p_1 = \frac{pT_1}{T} = p\left(\frac{460+t_1}{460+t}\right)$. So, also, by letting the temperature be constant, $T = T_1$ and $\frac{pV}{T} = \frac{p_1 V_1}{T}$, or $pV = p_1 V_1$, which is the same as formula **7**.

In formulas **7, 16, 17,** and **18,** it matters not with what units the pressures and volumes are measured, except that they must be the same throughout the same example, and the pressures must always be *absolute pressures.*

EXAMPLES FOR PRACTICE.

68. Solve the following:

1. A vessel contains 25 cubic feet of gas at a pressure of 18 pounds per square inch; if 125 cubic feet of gas having the same pressure are forced into the vessel, what will be the resulting pressure?

Ans. 108 lb. per sq. in.

2. A pound of air has a temperature of 126° and a pressure of 1 atmosphere; what volume does it occupy? Ans. 14.77 cu. ft.

3. A certain quantity of air has a volume of 26.7 cubic feet, a pressure of 19.3 pounds per square inch, and a temperature of 42°; what is the weight? Ans. 2.77 lb.

4. A receiver contains 180 cubic feet of gas at a pressure of 20 pounds per square inch; if a vessel holding 12 cubic feet be filled from the receiver until its pressure is 20 pounds per square inch, what will be the pressure in the receiver? Ans. $18\frac{2}{3}$ lb. per sq. in.

5. If 10 cubic feet of air having a pressure of 22 pounds per square inch and a temperature of 75° are heated until the temperature is 300°, the volume remaining the same, what is the new pressure?

Ans. 31.25 lb. per sq. in.

MIXTURES OF GASES.

69. If two liquids that do not act chemically on each other are mixed together and allowed to stand, it will be found that after a time the two liquids have separated, and that the heavier has fallen to the bottom. If two vessels of equal capacities containing gases of different densities are put in communication with each other, the gases will be found to have mixed in equal proportions after a short time. If one vessel is higher than the other, and the heavier gas is in the lower vessel, the same result will occur. The greater the difference of the densities of the two gases, the quicker they will mix. It is assumed that no chemical action takes place between the two gases. When the two gases have the same temperature and pressure, the pressure of the mixture will be the same; this is evident, since the total volume has not been changed, and unless the volume or temperature changes, the pressure cannot change. This property of the mixing of gases is a very valuable one, since, if they acted like liquids, carbonic-acid gas (the result of combustion), which is $1\frac{1}{2}$ times as heavy as air, would remain next to the earth instead of dispersing into the atmosphere, the result being that no animal life could exist.

70. Mixture of Equal Volumes of Gases Having Unequal Pressures.—*If two gases having equal volumes and temperatures, but different pressures, are mixed in a vessel whose volume is equal to one of the equal volumes of the gas, the pressure of the mixture will be equal to the sum of the two pressures, provided that the temperature remains the same as before.*

EXAMPLE.—Two vessels containing 3 cubic feet of gas, each at a temperature of 60° and subjected to pressures of 40 pounds and 25 pounds per square inch, respectively, are placed in communication with each other, and all the gas is compressed into one vessel. If the temperature of the mixture is also 60°, what is the pressure?

SOLUTION.—According to the rule just given, the pressure will be $40 + 25 = 65$ pounds per square inch. This may be proved by applications of Mariotte's law; thus, compress the gas whose pressure is 25 pounds per square inch, until its pressure is 40 pounds; its volume

may be found thus: $p\,v = p_1 v_1$, or $25 \times 3 = 40 \times v$; whence, $v = 1.875$ cubic feet. Let communication be established between the two vessels. The pressure will evidently be 40 pounds, and the total volume $3 + 1.875 = 4.875$ cubic feet. If this be compressed until the volume is 3 cubic feet, the temperature remaining at 60° throughout the whole operation, the final pressure may be found by formula **7**, $p\,v = p_1 v_1$. Thus, $40 \times 4.875 = p_1 \times 3$, and $p_1 = \frac{40 \times 4.875}{3} = 65$ pounds per square inch, as before.

71. Mixture of Two Gases Having Unequal Volumes and Pressures.

Let v and p = volume and pressure, respectively, of one of the gases;
v_1 and p_1 = volume and pressure, respectively, of the other gas;
V and P = volume and pressure, respectively, of the mixture.

Then, if the temperature remains the same,

$$VP = vp + v_1 p_1. \qquad (17.)$$

That is, *if the temperature is constant, the volume after mixture, multiplied by the resulting pressure, is equal to the volume of one gas before mixture multiplied by its pressure, plus the volume of the other gas multiplied by its pressure.*

EXAMPLE.—Two gases of the same temperature, having volumes of 7 cubic feet and $4\frac{1}{2}$ cubic feet, and whose pressures are 27 pounds and 18 pounds per square inch, respectively, are mixed together in a vessel whose volume is 10 cubic feet. The temperature of the two gases and of the mixture being 60° F., what is the resulting pressure?

SOLUTION.—Applying formula **17**,

$$PV = p\,v + p_1 v_1, \text{ or } P \times 10 = 27 \times 7 + 4\tfrac{1}{2} \times 18.$$

Hence, $$P = \frac{189 + 81}{10} = 27 \text{ lb. per sq. in.}$$ Ans.

72. Mixture of Two Volumes of Air Having Unequal Pressures, Volumes, and Temperatures.—If a body of air having a temperature t_1, a pressure p_1, and a volume v_1 is mixed with another volume of air having a temperature t_2, a pressure p_2, and a volume v_2, and the mixture has a volume V, pressure P, and a temperature t, then, either the new temperature t, the new volume V, or the new pressure P may be found, if the other two quantities are known, by the

following formula, in which T_1, T_2, and T are the absolute temperatures corresponding to t_1, t_2, and t:

$$PV = \left(\frac{p_1 v_1}{T_1} + \frac{p_2 v_2}{T_2}\right) T. \qquad \textbf{(18.)}$$

EXAMPLE.— 5 cubic feet of air having a tension of 30 pounds per square inch, and a temperature of 80° F., are required to be compressed together with 11 cubic feet of air having a tension of 21 pounds per square inch, and a temperature of 45° F., in a vessel whose cubical contents are 8 cubic feet. The new pressure is required to be 45 pounds per square inch. What is the temperature of the mixture?

SOLUTION.—Substituting in formula **18,**

$$45 \times 8 = \left(\frac{30 \times 5}{540} + \frac{21 \times 11}{505}\right) \times T, \text{ or } 360 = .7352\ T.$$

Hence, $T = \frac{360}{.7352} = 489.66°$, nearly, and $t = 29.66°$. Ans.

EXAMPLES FOR PRACTICE.

73. Solve the following:

1. Two vessels contain air at pressures of 60 and 83 pounds per square inch, respectively. The volume of each vessel is 8.47 cubic feet. If all of the air in both vessels is removed to another vessel, and the new pressure is 100 pounds per square inch, what is the volume of the vessel, the temperature being the same throughout? Ans. 12.11 cu. ft.

2. A vessel contains 11.83 cubic feet of air at a pressure of 33.3 pounds per square inch. It is desired to increase the pressure to 40 pounds per square inch by supplying air from a second vessel that contains 19.6 cubic feet of air at a pressure of 60 pounds per square inch. What will be the pressure in the second vessel after the pressure in the first has been raised to 40 pounds per square inch?

Ans. 55.96 lb. per sq. in.

3. If 4.8 cubic feet of air having a tension of 52 pounds per square inch and a temperature of 170° are mixed with 13 cubic feet of air having a tension of 78 pounds per square inch and a temperature of 265°, what must be the volume of the vessel containing the mixture, in order that the tension of the mixture may be 30 pounds per square inch and the temperature 80°? Ans. 32.31 cu. ft.

THE AIR PUMP.

74. The **air pump** is an instrument for removing air from an enclosed space. A section of the principal parts is shown in Fig. 11, and the complete instrument in Fig. 12.

The closed vessel R is called the *receiver*, and the space enclosed by it is that from which it is desired to remove the

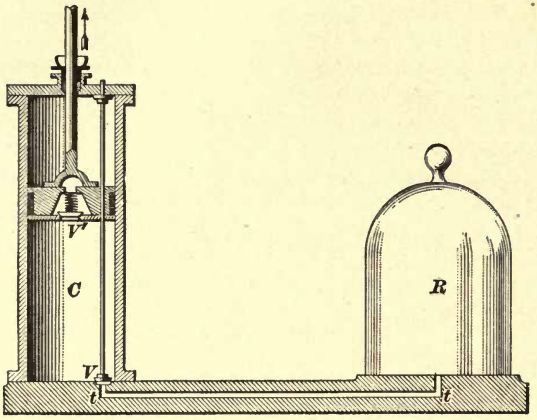

FIG. 11.

air. The receiver is usually made of glass, and the edges are ground so as to be perfectly air-tight. When made in the form shown, it is called a *bell-jar receiver*. The receiver rests upon a horizontal plate, in the center of which is an opening communicating with the pump cylinder C, by means of a bent tube t. The pump piston fits the cylinder accurately, and has a valve V' opening upwards. At the junction of the tube with the cylinder is another valve V also opening upwards. When the

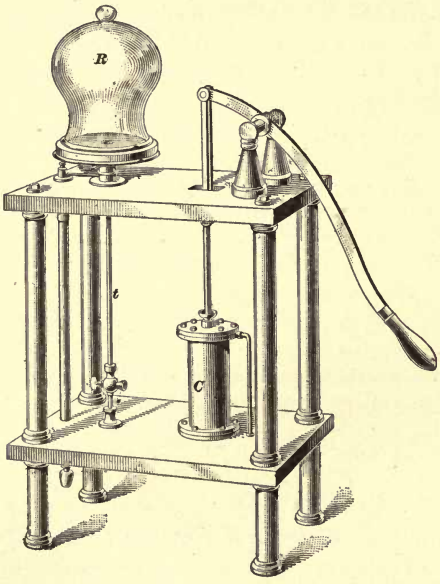

FIG. 12.

piston is raised the valve V' closes, and, since no air can get into the cylinder from above, the piston leaves a vacuum behind it. The pressure on top of V being now removed, the tension of the air in the receiver R causes V to rise; the air in the receiver then expands and occupies the space displaced by the piston, the space in the tube t and in the receiver R. The piston is now pushed down, the valve V closes, the valve V' opens, and the air in C escapes. The lower valve V is sometimes supported, as shown in Fig. 11, by a metal rod passing through the piston and fitting it somewhat tightly. When the piston is raised or lowered, this rod moves with it. A button near the upper end of the rod confines its motion to within very narrow limits, the piston sliding upon the rod during the greater part of the journey.

75. Degrees and Limits of Exhaustion.—Suppose that the volume of R and t together is four times that of C, and that there are, say, 200 grains of air in R and t, and 50 grains in C, when the piston is at the top of the cylinder. At the end of the first stroke, when the piston is again at the top, 50 grains of air in the cylinder C will have been removed, and the 200 grains in R and t will occupy the spaces R, t, and C. The ratio between the sum of the spaces R and t, and the total space $R+t\,C$ is $\frac{4}{5}$; hence, $200\times\frac{4}{5} = 160$ grains = the weight of air in R and t after the first stroke. After the second stroke, the weight of the air in R and t would be $200\times\frac{4}{5}\times\frac{4}{5} = 200\times(\frac{4}{5})^2 = 200\times\frac{16}{25} = 128$ grains. At the end of the third stroke the weight would be $[200\times(\frac{4}{5})^2]\times\frac{4}{5} = 200\times(\frac{4}{5})^3 = 200\times\frac{64}{125} = 102.4$ grains. At the end of n strokes the weight would be $200\times(\frac{4}{5})^n$. It is evident that *it is impossible to remove all of the air that is contained in R and t by this method.* It requires an exceedingly good air pump to reduce the tension of the air in R to $\frac{1}{50}$ of an inch of mercury. When the air has become rarefied to this extent, the valve V' will not lift, and, consequently, no more air can be exhausted.

76. Sprengel's Air Pump.—In Fig. 13, $c\,d$ is a glass tube longer than 30 inches, open at both ends, and connected,

by means of india-rubber tubing, with a funnel A filled with mercury and supported by a stand. Mercury is allowed to fall into this tube at a rate regulated by a clamp at c. The lower end of the tube $c\,d$ fits in the flask B, which has a spout at the side a little higher than the lower end of $c\,d$; the upper part has a branch at x to which a receiver R can be tightly fixed. When the clamp at c is opened, the first portions of the mercury that run out close the tube and prevent air from entering from below. These drops of mercury act like little pistons, carrying the air in front of them and forcing it out through the bottom of the tube. The air in R expands to fill the tube every time that a drop of mercury falls, thus creating a partial vacuum at R, which becomes more nearly complete as the process goes on. The escaping mercury falls into the dish H, from which it can be poured back into the funnel from time to time. As the exhaustion from R goes on, the mercury rises in the tube $c\,d$ until, when the exhaustion is complete, it forms a continuous column 30 inches high; in other words, it is a barometer whose Torricellian vacuum is

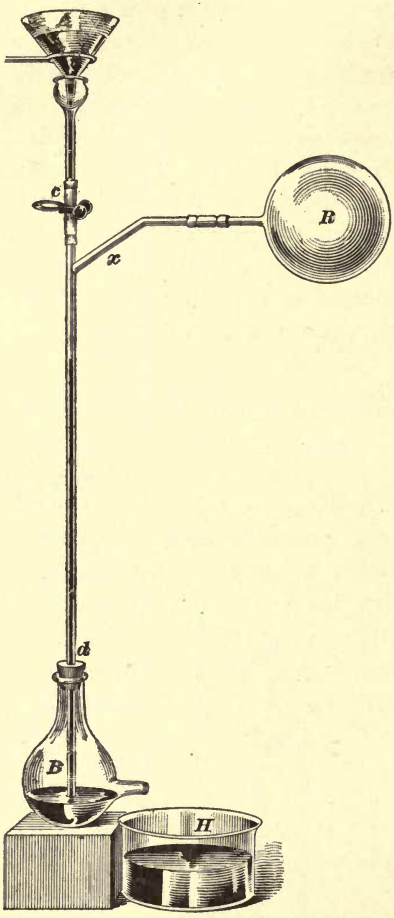

FIG. 13.

NOTE.—A theoretically perfect vacuum is sometimes called a *Torricellian vacuum*.

the receiver R. This instrument necessarily requires a great deal of time for its operation, but the results are very complete, a vacuum of $\frac{1}{43000}$ of an inch of mercury being sometimes obtained. By the use of chemicals in addition to the above, a vacuum of $\frac{1}{650000}$ of an inch of mercury has been obtained.

HEAT.

NATURE OF HEAT.

77. As to the exact nature of heat, scientists differ, but all modern thinkers and investigators agree that *heat is a form of energy*, and that it is a *kind of motion*. It is not our purpose to enter into the different theories regarding heat, but as much of the generally accepted theory will be given as will be necessary to make clear the principles that are to follow.

78. In Art. **3** it was stated that bodies are composed of molecules. Notwithstanding the extreme minuteness of the molecules, they play a very important part in the modern theory of heat. Each molecule attracts the molecules surrounding it in a manner similar to the attraction between earth and bodies near its surface, only with an immensely greater force in proportion to their sizes. Without going into any theory regarding the precise nature of heat, it will be taken for granted that each and every molecule has a rapid vibratory motion to and fro, and that the molecules are kept from getting beyond a certain distance from one another by the attractive force between them. This attractive force is called cohesion; without it, everything throughout the universe would instantly crumble into the finest dust.

In Art. **15** it was stated that the molecules are supposed to be round; it is likewise supposed that they are at a considerable distance apart, compared with their diameters. When heat is applied to a body, the number of the vibrations

is increased proportionally to the amount of heat supplied. In consequence of this increase, the distance through which a molecule moves is increased, and the force of cohesion that binds the molecules together is lessened. If enough heat is added to a solid, the force of cohesion is so far overcome that the body melts. If heat is supplied in sufficient quantity, the melted body becomes a vapor, and so long as it is kept at this temperature the force of cohesion has no effect, in consequence of the number of vibrations having been so far increased and the distance between any two molecules having become too great for the force of cohesion to act. If the vapor is cooled, the number of vibrations and also the distance between any two molecules will decrease; the force of cohesion begins to act and the body becomes a liquid. If cooled further and a sufficient quantity of heat is removed (in other words, if the number of vibrations is so far decreased that the molecules are comparatively near together), the body becomes a solid and remains so until the temperature is again increased to the melting point.

79. If a body is heated and brought near the hand, the sensation of warmth is felt; if heat is removed from this same body and it is again brought near the hand, the sensation of cold is felt. The heat that thus manifests itself is called *sensible heat*, because any change from any state to a hotter or colder state is indicated at once by the sense of feeling. The more sensible heat a body possesses, the hotter it is; the more sensible heat is taken away from it, the colder it is.

TEMPERATURE.

80. The most common heat measure we have is that which we gain by means of the sensation of warmth it produces. According to the character of this sensation a body is said to be cold, warm, or hot. These terms all refer to the power that one body has of communicating heat to other bodies. The measure of this power is termed

temperature, which may be more exactly embodied in the following definition: *The temperature of a body is a measure of the intensity of its heat, and is further defined as the thermal state of a body considered with reference to its power of communicating heat to other bodies.*

81. For scientific purposes, the sensations are not sufficiently accurate methods of measuring temperature; accordingly, temperature is usually measured by certain of the effects produced by heat. Among these, one of the most convenient is, that most bodies expand with a rise of temperature. This expansion is distinctly perceptible in solids; it occurs to a greater extent in liquids, and most of all with gases. For the general purposes of temperature measurement, mercury and alcohol are the most convenient substances, the former because it boils only at a very high temperature, and the latter because it does not solidify at the greatest known cold produced by ordinary means. One of these liquids, enclosed in a suitable vessel, constitutes the temperature-measuring instrument termed a *thermometer*.

82. In constructing a thermometer, a bulb is blown at one end of a glass tube of a very narrow bore; the bulb and a portion of this tube are next filled with carefully purified mercury; this is boiled, and thus all air and moisture is driven out of the tube; the open end is then hermetically sealed by fusing the glass itself. The bulb and a portion of the tube are thus filled with mercury and the remainder of the tube is a vacuum, save for the presence of a minute quantity of mercury vapor. On heating the bulb of this instrument the mercury expands and rises considerably in the stem. Heat has a tendency to so distribute itself throughout any body or series of bodies as to make them of the same temperature; consequently, if the thermometer is placed in contact with the body whose temperature it is desired to measure, a redistribution of heat occurs until the two are at the same temperature. That is to say, if the body is the colder it receives heat from the thermometer, and if

it is hotter it yields heat to the thermometer, until the temperature of the two is the same. The two being in efficient contact, this stage is indicated by the mercury becoming stationary in the thermometer tube. Now the mercury is constant for any one temperature; therefore, to register temperature, it is only necessary to have, further, a scale or series of graduations attached to the stem of the instrument, by which the temperature may always be read.

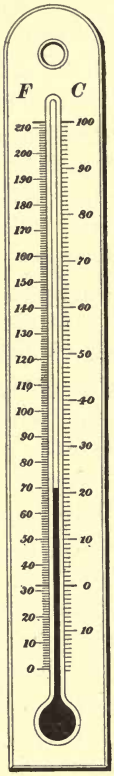

FIG. 14.

83. In Fig. 14 is shown a mercurial thermometer with two sets of graduations on it. The one on the left, marked *F*, is the *Fahrenheit* thermometer, so named after its inventor, and is the one commonly in use in this country and in England; the one on the right, marked *C*, is the *centigrade* thermometer, and is used by scientists throughout the world on account of the graduations being better adapted for calculations.

84. In graduating thermometers two fixed points of temperature are almost universally employed. These are the temperatures of melting ice, and of the steam from boiling water. Certain simple precautions being taken, these temperatures are always constant. It is, in addition, necessary to graduate the thermometers, so as to register temperatures intermediate between these two points, and also below and above them. The most convenient system of graduation is that of *Celsius*, known as the *centigrade scale*. In this scale the distance between the melting point of ice (or the freezing point of water, as it is commonly called) and the temperature of steam at atmospheric pressure is divided into 100 equal graduations or degrees. (This latter temperature is more commonly described as being that of the boiling point of water.) The freezing point is called zero, or 0°, and the boiling point 100°. Degrees of the same

value are carried above and below the boiling and freezing points, respectively; the temperature below 0° is considered negative, and is counted downwards from zero; thus, 10° below freezing point is −10°, and so on. Degrees above the boiling point are simply counted upwards from zero; thus, 10° above the boiling point is 110°, and so on.

85. In graduating Fahrenheit thermometers, the distance between the freezing and boiling points is divided into 180 equal parts, and degrees of the same size are carried above and below the boiling and freezing points. Fahrenheit assumed that the greatest cold attainable was 32° below the freezing point, and accordingly took that point as his zero and reckoned from it upwards. The freezing point thus became 32° F. and the boiling point $32 + 180 = 212°$. Degrees below the Fahrenheit zero are reckoned downwards from zero and are considered as negative.

In Germany and Russia temperature is reckoned on the Réaumur scale, in which the freezing point is 0° and the boiling point is 80°.

86. Of these three thermometers, the centigrade is used the most, but since the Fahrenheit instrument is the one in general use in this country, all temperatures in this section will be understood to be in Fahrenheit degrees, unless otherwise stated.

87. It is frequently necessary to be able to compare these scales, and to translate temperatures from any one into another. For example, what would 80° C. be on the Fahrenheit scale?

Since the number of degrees between the freezing point and the boiling point on the centigrade scale is 100, and on the Fahrenheit 180, it is evident that, if F = number of degrees Fahrenheit, and C = number of degrees centigrade,

$$F : C :: 180 : 100, \text{ or } F = \tfrac{180}{100}\, C = \tfrac{9}{5}\, C.$$

Also,
$$C = \tfrac{100}{180}\, F = \tfrac{5}{9}\, F.$$

Therefore, to change centigrade temperatures into their corresponding Fahrenheit values:

Rule.—*Multiply the temperature, centigrade, by $\frac{9}{5}$ and add 32°; the result will be the temperature, Fahrenheit.*

To change Fahrenheit temperatures into their corresponding centigrade values:

Rule.—*Subtract 32° from the temperature, Fahrenheit, multiply by $\frac{5}{9}$, and the result will be the temperature, centigrade.*

Expressing these two rules by means of formulas,

Let t_c = temperature centigrade;
t_f = temperature Fahrenheit.

Then, $t_f = \frac{9}{5} t_c + 32°$, **(19.)**

and $t_c = \frac{5}{9}(t_f - 32°)$. **(20.)**

EXAMPLE 1.—Change (*a*) 100° C., (*b*) 4° C., and (*c*) −40° C. into Fahrenheit temperatures.

SOLUTION.—(*a*) $t_f = \frac{9}{5} t_c + 32 = \frac{9}{5} \times 100 + 32 = 212°$ F. Ans.
(*b*) $t_f = \frac{9}{5} \times 4 + 32 = 39.2°$ F. Ans.
(*c*) $t_f = \frac{9}{5} \times -40 + 32 = -40°$ F. Ans.

EXAMPLE 2.—Change (*a*) 60° F., (*b*) 32° F., and (*c*) −20° F. into their corresponding centigrade temperatures.

SOLUTION.—(*a*) $t_c = (t_f - 32)\frac{5}{9} = (60 - 32)\frac{5}{9} = 15\frac{5}{9}°$ C. Ans.
(*b*) $t_c = (32 - 32)\frac{5}{9} = 0°$ C. Ans.
(*c*) $t_c = (-20 - 32)\frac{5}{9} = -28\frac{8}{9}°$ C. Ans.

88. The absolute temperature is the temperature measured above the point of absolute zero (see Art. **64**). Hence, on the Fahrenheit scale the absolute temperature T is $460° + t°$ when t = the ordinary temperature, and is above zero. If $t°$ is below zero, its value is negative, and the absolute temperature T is $460° + (-t°) = 460° - t°$.

Throughout this section, where temperatures are mentioned, t will denote the ordinary temperature indicated by the thermometer, and T, the absolute temperature.

EXAMPLE.—What are the absolute temperatures corresponding to the Fahrenheit temperatures 212°, 32°, and −39.2°?

SOLUTION.— $460° + 212° = T = 672°$ F. Ans.
$460° + 32° = T = 492°$ F. Ans.
$460° - 39.2° = T = 420.8°$ F. Ans.

The absolute temperature on the centigrade scale is $T = 273\frac{1}{3}° + t°$ when $t°$ is above zero, or $T = 273\frac{1}{3}° - t°$ when $t°$ is below zero.

EXAMPLE.—What are the absolute temperatures corresponding to 100° C., 4° C., and −40° C.?

SOLUTION.— $273\frac{1}{3}° + 100° = T = 373\frac{1}{3}°$ C. Ans.
$273\frac{1}{3}° + 4° = T = 277\frac{1}{3}°$ C. Ans.
$273\frac{1}{3}° - 40° = T = 233\frac{1}{3}°$ C. Ans.

EXAMPLES FOR PRACTICE.

89. 1. What are the absolute temperatures corresponding to (*a*) 120° R.? (*b*) 120° C.? (*c*) 120° F.?

Ans. (*a*) $338\frac{2}{3}°$ R.; (*b*) $393\frac{1}{3}°$ C.; (*c*) 580° F.

2. Change −10° R. to the corresponding Fahrenheit and centigrade readings. Ans. $9\frac{1}{2}°$ F.; $-12\frac{1}{2}°$ C.

EXPANSION OF BODIES BY HEAT.

90. The volume of any body—solid, liquid, or gaseous—is always changed if the temperature is changed; nearly all bodies expand when heated, and contract when cooled. In solids having definite figures, the expansion may be considered in three ways, according to the conditions.

1. The expansion in one direction, as the elongation of an iron bar; this is called *linear* expansion.

2. *Surface* expansion, where the area is increased.

3. *Cubical* expansion, where the increase in the whole volume is considered.

91. In Fig. 15 is shown an apparatus for exhibiting the linear expansion of a solid body. A metal rod *A* is fixed at one end by a screw *B*, the other end passing freely through

the eye C, held in the post, and pressing against the short arm of the indicator F. The rod is heated as shown, and its elongation causes the indicator to move along the arc $D\,E$.

An illustration of surface expansion is afforded nearly every day in machine shops, particularly in locomotive

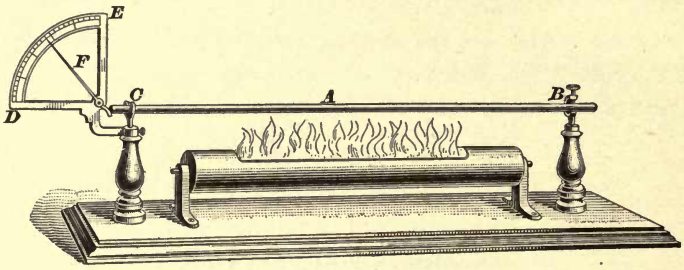

FIG. 15.

shops, where piston rods, crankpins, etc. are shrunk in, and tires are shrunk on their centers. In shrinking a tire on, it is bored a little smaller than the wheel center. The tire is then heated until the area of its circumference is expanded enough to allow it to slide over the wheel center. On being

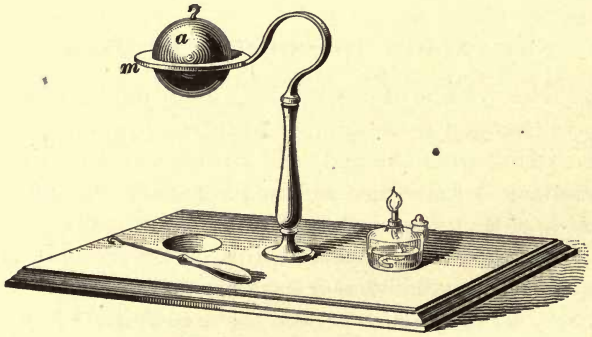

FIG. 16.

cooled with water, the tire contracts, and tends to regain its original area, but is prevented by reason of the wheel center being a trifle larger. This causes the tire to hug the center with immense force, and prevents it from coming off.

Cubical expansion may be illustrated by means of a *Gravesandes' ring.* This consists of a brass ball *a*, Fig. 16, which at ordinary temperatures passes freely through the ring *m*, of very nearly the same diameter. When the ball is heated it expands so much that it will no longer pass through the ring.

The expansion of liquids is clearly shown in the mercurial and alcohol thermometers. The expansion of gases was treated to some extent in Arts. **56** to **72.**

92. Coefficient of Expansion.—Suppose that the temperature of the metal rod shown in Fig. 15 was 32° F. before heating, and that its length at that temperature is exactly 10 feet. It is found that when the temperature is raised 1°, or to 33°, the bar is 10 feet $+\frac{1}{1200}$ inch long. The linear expansion is then (10 feet $+\frac{1}{1200}$ inch) $-$ 10 feet $=\frac{1}{1200}$ inch, and the ratio between this expansion and the original length of the bar is

$$\frac{1}{1{,}200} : 10 \times 12 = \frac{1}{1{,}200 \times 120} = .000006944.$$

For every increase of temperature of 1°, this rod elongates .000006944 of its length. This number .000006944, which is equal to the expansion of the rod for 1° rise of temperature divided by the original length, is called the *coefficient of linear expansion.* Had the temperature of the rod been increased 100° instead of 1°, the amount of elongation would have been $.000006944 \times 100 = .0006944$ of its length, or $.0006944 \times 120 = .083328$ inch, or $\frac{1}{12}$ inch.

93. The table following contains the coefficients of expansion for a number of solids, mercury, and alcohol, and the average cubical expansion of gases. No liquids are given except mercury and alcohol, for the reason that the coefficient of expansion for liquids is different at different temperatures.

TABLE 2.

TABLE OF COEFFICIENTS OF EXPANSION.

Name of Substance.	Linear Expansion.	Surface Expansion.	Cubical Expansion.
Cast iron	.00000617	.00001234	.00001850
Copper	.00000955	.00001910	.00002864
Brass	.00001037	.00002074	.00003112
Silver	.00000690	.00001390	.00002070
Bar iron	.00000686	.00001372	.00002058
Steel (untempered)	.00000599	.00001198	.00001798
Steel (tempered)	.00000702	.00001404	.00002106
Zinc	.00001634	.00003268	.00004903
Tin	.00001410	.00002820	.00003229
Mercury	.00003334	.00006668	.00010010
Alcohol	.00019259	.00038518	.00057778
Gases			.00203252

94. Let L = length of any body;
l = amount of expansion or contraction due to heating or cooling the body;
A = area of any section of the body;
a = increase or decrease of area of the same section after heating or cooling the body;
V = volume of the body;
v = increase or decrease in volume due to heating or cooling the body;
C_1 = coefficient of expansion taken from column 1, Table 2;
C_2 = coefficient taken from column 2, Table 2;
C_3 = coefficient taken from column 3, Table 2;
t = difference in degrees of temperature between the original temperature and the temperature of the body after it has been heated or cooled.

Then, $$l = L C_1 t. \qquad (21.)$$

$$a = A C_2 t. \qquad (22.)$$

$$v = V C_3 t. \qquad (23.)$$

EXAMPLE.—How much will a bar of untempered steel 14 feet long expand, if its temperature is raised 80° ?

SOLUTION.—Since only one dimension is given, that of length, linear expansion only can be considered. From Table 2, the coefficient of linear expansion per unit of length for a rise in temperature of 1° is found to be .00000599 for untempered steel. Hence, using formula **21**, $l = L C_1 t$, and substituting, $14 \times .00000599 \times 80 = .0067088$ foot, or $.0067088 \times 12 = .0805056$ inch. Ans.

95. This expansion seems very small, but in engineering constructions, when long pieces are rigidly connected, it must be taken into account. If the cross-section of the above bar was 2 inches square, and the bar was fitted tightly between two supports, an expansion of the above amount would exert a pressure against the supports of about 58,000 pounds.

Suppose that an iron rod $1\frac{1}{2}$ inches in diameter and 100 feet long were used as a tie-rod in constructing a bridge; that it was put in place and securely fastened to two rigid supports during a warm day in summer when the temperature in the sunlight was, say, 110°. On a cold day in winter when the thermometer registered zero, the amount that the bar would tend to shorten owing to this change in temperature would be, substituting these values in formula **21**,

$$.00000686 \times 100 \times 110 = .07546 \text{ foot} = .90552 \text{ inch.}$$

If this rod were rigidly secured so that it could neither stretch nor shorten, it would then exert a pull on the supports of about 33,400 pounds.

EXAMPLE 1.—The wheel center of a locomotive driver is turned to exactly 50 inches in diameter. If the steel tire be bored 49.94 inches in diameter, to what temperature must the tire be raised in order that it may be easily shoved over the center ? Assume that the diameter of the tire is expanded to $\frac{1}{1000}$ of an inch larger than the center, and that the original temperature is 60°.

SOLUTION.—For this case, formula **22** may be used. The original diameter of the tire is 49.94 inches, and it is to be increased to 50.001 inches. The area of a circle 49.94 inches in diameter is 1,958.79 square inches. Area of circle 50.001 inches in diameter is 1,963.58 square inches. The difference between them is $1{,}963.58 - 1{,}958.79 = 4.79$ square inches $= a$ in formula **22**. Hence, since $C_2 = .00001198$, and $A = 1{,}958.79$, substitute these values in $a = A\,C_2\,t$, and $4.79 = 1{,}958.79 \times .00001198 \times t = .023466\,t$. Therefore, $t = \frac{4.79}{.023466} = 204.125°$, and $204.125° + 60° = 264.125°$. Ans.

NOTE.—Owing to the form of the equation here denoted by formula **22**, and to the manner in which the coefficients C_2 were determined, this example may be more easily solved by means of formula **21**. Thus, regard the diameter as a linear dimension and apply formula **21**. Increase in diameter $= l = 50.001 - 49.94 = .061$ inches. $L = 49.94$ and $C_1 = .00000599$. Substituting, $.061 = 49.94 \times .00000599 \times t$, or,

$$t = \frac{.061}{49.94 \times .00000599} = 203.92°. \quad \text{Ans.}$$

The slight difference in the two answers is immaterial, and is to be expected.

EXAMPLE 2.—What is the decrease in volume of a copper cylinder 30 inches long and 22 inches in diameter, if cooled from 212° to 0°, the measurement being taken at a temperature of 70°?

SOLUTION.— $212° - 70° = 142°$ = the increase in temperature above 70°. Use formula **23**, $v = V\,C_3\,t$.

$V = 22^2 \times .7854 \times 30 = 11{,}404$ cu. in.
$v = 11{,}404 \times .00002864 \times 142 = 46.38$ cu. in.
$11{,}404 + 46.38 = 11{,}450.38$ cu. in. = the volume at 212°.
$70° - 0° = 70°$ = the difference in temperature.
$v = 11{,}404 \times .00002864 \times 70 = 22.86$ cu. in., nearly.
$46.38 + 22.86 = 69.24$ cu. in. Ans.

96. The bars of a furnace must not be fitted tightly at their extremities, but must be free at one end, otherwise in expanding they will split the masonry.

In laying the rails on railroads, a small space is left between the successive rails; for, if they touched, the force of expansion would cause them to curve. Water pipes are fitted to one another by means of telescope joints, which allow room for expansion; so, also, are steam pipes, by means of the so called expansion joints. If a glass vessel is heated or cooled too rapidly, it cracks, especially if it is thick; the reason for this is that since glass is a poor conductor of heat,

the sides become unequally heated, and, consequently, unequally expanded, which causes a fracture.

97. It will be found on trial that the three preceding formulas will not work back; i. e., if the length of a bar after it has been heated is found by formula **21**, and an attempt is made to reduce the bar to its original length by again applying formula **21** and substituting for t the same value as in the first case, the value obtained for l will be slightly different in the two cases. The difference, however, is so slight that it is neglected in practice. If, however, the student desires to obtain exactly the same result in both cases, he must use the following more cumbersome formula, in which t_1, t_2, l_1, l_2 are, respectively, the original and final temperatures, the original and final lengths, and C_1 has the same value as in formula **21**:

$$l_2 = \left[\frac{1 + C_1(t_2 - 32)}{1 + C_1(t_1 - 32)}\right] l_1. \qquad (24.)$$

This formula is always used when calculating the expansion of gases by substituting v_1, v_2, and $\frac{1}{492}$ for l_1, l_2, and C_1, respectively. The result obtained will be exactly the same as those obtained by formula **12**. For, substituting the values as directed, the formula becomes

$$v_2 = \left[\frac{1 + \frac{1}{492}(t_2 - 32)}{1 + \frac{1}{492}(t_1 - 32)}\right] v_1 = \frac{\frac{492 + t_2 - 32}{492}}{\frac{492 + t_1 - 32}{492}} \times v_1$$

$$= \left(\frac{460 + t_2}{460 + t_1}\right) v_1.$$

98. Although, as stated previously, the expansion of solids and liquids is very nearly uniform throughout all ranges of temperature, water is a marked exception to the general rule. If water is cooled down from its boiling point, it continually contracts until it reaches 39.2° F., when it begins to expand, until it freezes at 32° F. On the other hand, if water at 32° F. is heated, it contracts until it reaches 39.2° F., when it commences to expand. Therefore, the

density of water is greatest where this change occurs. The importance of this exception is seen in the fact that ice forms on the *surface* of the water, since it is lighter than the warmer body of water lying at varying depths below it. Were it not for this fact, all the large bodies of water would freeze solid and would so affect the earth's climate that it would be uninhabitable. The coefficient of expansion of water is such a very changeable quantity (varying with the temperature) that a special table is necessary.

99. The effect of heat on the expansion of gases was treated in Arts. **61** to **67**, and will not be repeated here. It should be stated, however, that the constant .37052 used in formulas **14** and **15** has that value for air only. For other gases it varies. If the value of this constant for any gas be represented by R, formula **15** becomes

$$pV = RWT. \qquad (25.)$$

100. The value R for several gases is given in the following table:

TABLE 3.

Gas.	Volume of 1 Lb. at 32° F. and a Tension of 1 Atmosphere (14.7 Lb. per Sq. In.). Cubic Feet.	Weight of 1 Cu. Ft. at 32° F. and a Tension of 1 Atmosphere (14.7 Lb. per Sq. In.). Pounds.	R.
Air	12.3880	.08073	.37052
Oxygen	11.2056	.08925	.33552
Nitrogen	12.7226	.07860	.38143
Hydrogen	178.8910	.00559	5.34946

EXAMPLE 1.—What is the volume of 3 ounces of hydrogen gas having a tension of 20 pounds per square inch and a temperature of 80° ?

SOLUTION.— 3 ounces $= \frac{3}{16}$ of a pound. Since $t = 80°$, $T = 460 + 80 = 540°$, $R = 5.34946$ from Table 3. Hence, by formula **25**,

$$pV = RWT, \text{ or } 20\,V = 5.34946 \times \tfrac{3}{16} \times 540 = 541.6328,$$

and $$V = \frac{541.6328}{20} = 27.08164 \text{ cu. ft.}; \text{ say, } 27.082 \text{ cu. ft. Ans.}$$

EXAMPLE 2.—What is the weight of 10 cubic feet of oxygen having a tension of 1 atmosphere and a temperature of 60° ?

SOLUTION.—By formula **25**,

$$pV = RWT, \text{ or } 10 \times 14.7 = .33552 \times W \times 520.$$

Hence, $147 = 174.4704\ W$, and $W = \frac{147}{174.4704} = .84255$ lb. Ans.

In Table 2 the coefficient of expansion for gases was given as .00203252; this is the fraction $\frac{1}{492}$ reduced to a decimal. This value of the coefficient of expansion is very nearly the same for all gases, particularly for those that are very difficult to liquefy.

EXAMPLES FOR PRACTICE.

101. Solve the following:

1. (*a*) How much will an iron tie-rod 60 feet long expand when the temperature is raised from 40° to 110° ? (*b*) Calculate also by formula **24**. (*c*) What is the difference in the two results ?

Ans. (*a*) .345744 in. (*b*) .345725 in. (*c*) .000019 in.

2. To what temperature must a steel tire of 59.93 inches internal diameter be raised in order that its diameter may be 60.0015 inches ? Original temperature = 71°. Ans. 270°.

3. What is the volume of .68 pound of nitrogen gas having a tension of 20 pounds per square inch and a temperature of 345° ?

Ans. 10.44 cu. ft.

HEAT PROPAGATION.

102. Conduction of Heat.—Conduction is the slow progress of the vibratory motion from places of higher to places of lower temperature in the *same* body. Different bodies possess very different conducting powers, the good conductors being those in which conduction is most rapid, and the bad conductors those in which it is very slow. Experiments have established a numerical comparison of the conducting powers of many bodies. Representing the conducting power of silver by 100, the following table shows

the conducting power of a number of other metals as determined by Wiedemann and Franz:

TABLE 4.

Silver	100.0	Iron	11.9
Copper	73.6	Steel	11.6
Gold	53.2	Lead	8.5
Brass	23.1	Platinum	8.4
Zinc	19.0	Rose's alloy	2.8
Tin	14.5	Bismuth	1.8

103. As a class, the metals are the best conductors; the fluids, both liquid and gaseous, are very poor conductors, their conducting power being hardly appreciable. Water, for example, can be made to boil at the top of a vessel, while a cake of ice is fastened within a few inches of the surface. If thermometers are placed at different depths, *while water boils at the top*, it is found that the conduction of heat downwards is very slight.

Organic substances conduct heat poorly. This enables trees to withstand great and sudden changes in the atmosphere without injury. The bark is a poorer conductor than the wood beneath it. Cotton, wool, straw, bran, etc. are all poor conductors. Rocks and earth are poorer conductors, the less dense and homogeneous the mass; hence, the length of time required for the sun's rays to penetrate the earth. The mean highest temperature of the air near the ground in Central Europe is in the month of July, but at a depth of from 25 to 28 feet in the earth the highest temperature occurs in the month of December.

104. Dynamical Theory of Heat.—Before going any further, it will be convenient to explain here the view now generally taken as to the mode in which heat is propagated. On this subject, it is stated in Ganot's Physics, "A hot body is one whose molecules are in a state of vibration. The higher the temperature of a body, the more rapid are these

vibrations, and a diminution in temperature is but a diminished rapidity of the vibration of the molecules. The propagation of heat through a bar is due to a gradual communication of this vibratory motion from the heated part to the rest of the bar. A good conductor is one that readily takes up and transmits the vibratory motion from molecule to molecule, while a bad conductor is one that takes up and transmits the motion with difficulty. But even through the best of the conductors, the propagation of this motion is comparatively slow. How, then, can be explained the instantaneous perception of heat when a screen is removed from a fire, or when a cloud drifts from the face of the sun? In this case, the heat passes from one body to another without affecting the temperature of the medium that transmits it. In order to explain these phenomena, it is imagined that all space, the space between the planets and the stars, as well as the interstices in the hardest crystal and the heaviest metal—in short, matter of any kind—is permeated by a medium having the properties of matter of infinite tenuity, called *ether*. The molecules of a heated body being in a state of intensely rapid vibration, communicate their motion to the ether around them, throwing it into a system of waves, which travel through space and pass from one body to another with the velocity of light. When the undulations of the ether reach a given body, the motion is given up to the molecules of that body, which, in their turn, begin to vibrate; that is, the body becomes heated. The process of this motion through the ether is termed radiation, and what is called a ray of heat is merely one series of waves moving in a given direction.

HEAT MEASUREMENT.

THERMAL UNITS.

105. Quantity of Heat.—Temperature is not a measure of quantity of heat, for a thermometer would indicate the same temperature both in a vessel containing a pint, and one containing a gallon, of boiling water, although it is

evident that the one must contain eight times as much heat as the other; further, to raise the gallon of water to the boiling point, eight times the amount of heat necessary to similarly raise the pint is required. This leads us to the mode of measuring and indicating *quantity* of heat. *Quantity of heat is measured by the amount necessary to raise a certain weight of some body from one temperature to another fixed temperature.*

106. There are three units in use for measuring the quantity of heat:

1. The **British thermal unit,** which represents the amount of heat required *to raise 1 pound of water 1° Fahrenheit.* Instead of writing out the words "British thermal unit" in full, it is customary to abbreviate them to B. T. U. Thus, 7 pounds of water raised 15° F., would equal $7 \times 15 = 105$ B. T. U. The unit of heat used in this section will be the British thermal unit.

2. The **thermal unit,** which represents the amount of heat that is necessary *to raise 1 pound of water 1° centigrade.* Since 1° C. $= \frac{9}{5} \times 1°$ F., it follows that the thermal unit is $\frac{9}{5}$ times as large as a B. T. U. Hence, to change B. T. U. into thermal units, multiply the number of B. T. U. by $\frac{5}{9}$. To change thermal units into B. T. U., multiply the number of thermal units by $\frac{9}{5}$.

3. The **calorie,** which represents the amount of heat necessary *to raise 1 kilogram of water 1° centigrade.* One kilogram = 2.2 pounds and 1° C. $= \frac{9}{5} \times 1°$ F.; hence, a calorie $= 2.2 \times \frac{9}{5} = 3.96$ B. T. U. The calorie is used in Germany, France, and in other countries using the metric system of weights and measures.

SPECIFIC HEAT.

107. When equal weights of two different substances having the same temperature are placed in similar vessels and subjected for the same length of time to the heat of the same lamp, or are placed at the same distance in front of the same fire, it is found that their final temperatures will differ

considerably; thus, mercury will be much hotter than water. But, as, from the conditions of the experiment, they have each been receiving the same amount of heat, it is clear that the quantity of heat sufficient to raise the temperature of mercury through a certain number of degrees will raise the same weight of water through a less number of degrees; in other words, it requires more heat to raise a certain weight of water 1° than it does to raise the same weight of mercury 1°. Conversely, if the same quantities of water and of mercury at 200° are allowed to cool down to the temperature of the room, the water will require a much longer time for the purpose than the mercury; hence, in cooling through the same number of degrees, water gives up more heat than mercury.

The number of heat units or parts of a heat unit required to raise the temperature of a given quantity of any substance 1° is called the **specific heat** *of that substance.* It will be seen from the above definition, that *the specific heat of a substance is the ratio between the amount of heat required to raise the temperature of the substance 1° and the amount of the heat required to raise the temperature of the same weight of water 1°.*

108. If the specific heat of lead were given as .0314, it would mean that the amount of heat required to raise a certain weight of lead 1° would raise the same weight of water only .0314 of 1°; or it would mean that .0314 B. T. U. would raise the temperature of 1 pound of lead 1° F.

EXAMPLE.—The specific heat of copper is .0951; how many B. T. U. will it take to raise the temperature of 75 pounds 180°?

SOLUTION.—Since it takes .0951 B.T.U. to raise 1 pound of copper 1°, it will take $.0951 \times 75 \times 180$ to raise 75 pounds 180°. Hence,

$$.0951 \times 75 \times 180 = 1{,}283.85 \text{ B.T.U. Ans.}$$

In the example just given, if it had been required to raise 75 pounds of water 180° (that is, from the freezing point to the boiling point), it would have taken $75 \times 180 = 13{,}500$ B. T. U., and $\frac{1{,}283.85}{13{,}500} = .0951 =$ the specific heat of copper.

109. The following is the formula for finding the number of B. T. U. required to raise the temperature of a substance a given number of degrees, or for finding the number of B. T. U. given up by a body in cooling a given number of degrees.

Let W = weight of body in pounds;
s = specific heat of the substance composing the body;
t = original temperature of body;
t_1 = final temperature of body;
n = number of B. T. U. required, or given up, in changing the temperature of the body from $t°$ to $t_1°$.

Then, $$n = W(t_1 - t)\, s. \qquad (26.)$$

EXAMPLE.—A piece of wrought iron weighing 31.3 pounds and having a temperature of 900° is cooled to a temperature of 60°; how many units of heat does it give up? The specific heat of wrought iron is .1138.

SOLUTION.—Apply formula **26**, $n = W(t_1 - t)\, s.$ Substituting,
$n = 31.3 \times (900 - 60) \times .1138 = 2{,}992.03$ B. T. U. Ans.

If a body be cooled from a temperature t down to a temperature t_1, the value of n will be negative, the minus sign indicating that the body was cooled.

110. In the following table are given the specific heats of a number of substances under constant pressure:

TABLE 5.

SOLIDS.

Copper	0.0951	Cast iron	0.1298
Gold	0.0324	Lead	0.0314
Wrought iron	0.1138	Platinum	0.0324
Steel (soft)	0.1165	Silver	0.0570
Steel (hard)	0.1175	Tin	0.0562
Zinc	0.0956	Ice	0.5040
Brass	0.0939	Sulphur	0.2026
Glass	0.1937	Charcoal	0.2410

LIQUIDS.

Water	1.0000	Lead (melted)	0.0402
Alcohol	0.7000	Sulphur (melted)	0.2340
Mercury	0.0333	Tin (melted)	0.0637
Benzine	0.4500	Sulphuric acid	0.3350
Glycerine	0.5550	Oil of turpentine	0.4260

GASES.

Gases.	Constant Pressure.	Constant Volume.
Air	0.23751	0.16847
Oxygen	0.21751	0.15507
Nitrogen	0.24380	0.17273
Hydrogen	3.40900	2.41226
Superheated steam	0.48050	0.34600
Carbonic oxide	0.24790	0.17580
Carbonic acid	0.40400	0.15350

111. The reason that there are two values for the specific heat of gases is that it takes less heat to raise the temperature of a gas when the volume is constant than when the pressure is constant, but the volume varies. Thus, consider a closed cylinder filled with gas. If heat is applied, the pressure and temperature will both increase. Denoting the specific heat at constant pressure by s_p and at constant volume by s_v, the number of heat units required to heat the gas from $t°$ to $t_1°$ will be $s_v W(T_1 - T)$. If, however, the cylinder is imagined to be fitted with a frictionless piston, free to move up or down, and heat be applied, the gas will expand, overcoming a resistance equal to the weight of the piston plus the pressure of the atmosphere. Hence, in addition to the heat required to increase the vibratory movement of the molecules, heat is also required to overcome the outer pressure, which remains constant in this case. The number of heat units necessary will then be $s_p W(T_1 - T)$.

112. Mixing Two Bodies of Unequal Temperatures. If a certain quantity of water having a temperature of 40° is mixed with a like quantity having a temperature of 100°, it is evident that the temperature after mixing will be $\frac{40+100}{2}$ = 70°. But if 5 pounds of water having a temperature of 40° is mixed with 5 pounds of copper having a temperature of 100°, the temperature after mixing will not be 70°. The resulting temperature may be found by the following formula, provided there is no change of the state in the body (ice melting into water, etc.):

$$t = \frac{W_1 s_1 t_1 + W_2 s_2 t_2 + W_3 s_3 t_3 + \text{etc.}}{W_1 s_1 + W_2 s_2 + W_3 s_3 + \text{etc.}}, \qquad (27.)$$

in which t is the final temperature of the mixture; W_1, s_1, and t_1, the weight, specific heat, and temperature, respectively, of one body; W_2, s_2, and t_2, the same for second body; and W_3, s_3, and t_3, the same for a third body, etc.

Remembering that the specific heat of water is 1, and getting the specific heat of copper from Table 5, the temperature t will be

$$t = \frac{5\times1\times40+5\times.0951\times100}{5\times1+5\times.0951} = 45.21°, \text{ nearly.}$$

EXAMPLE 1.—If 21 pounds of water at a temperature of 52° are mixed with 40 pounds of water at a temperature of 160°, what is the temperature of the mixture?

SOLUTION.—Since the specific heat of water is 1, it may be left out in applying formula **27**, and the temperature is found to be

$$t = \frac{21\times52+40\times160}{21+40} = 122.82°. \quad \text{Ans.}$$

EXAMPLE 2.—A copper vessel weighing 2 pounds is partly filled with water having a temperature of 80° and weighing 7.8 pounds. A piece of wrought iron weighing 3 pounds 4 ounces and having a temperature of 780° is dropped into this water. What is the final temperature of the mixture?

SOLUTION.—Substituting the values given in formula **27**, and remembering that the original temperatures of the copper vessel and the water that it contains are the same (3 lb. 4 oz. = 3.25 lb.), we have

$$t = \frac{2\times.0951\times80+7.8\times80+3.25\times.1138\times780}{2\times.0951+7.8+3.25\times.1138} = 110.97°, \text{ nearly.}$$

Ans.

EXAMPLE 3.—A wrought-iron ball weighing 1 pound is placed in a reheating furnace: when it has attained the temperature of the furnace, it is taken out and placed in a copper vessel weighing $\frac{1}{2}$ pound and containing exactly 2 pounds of water at a temperature of 75°. Assuming that no water escapes as steam, and that the temperature of the ball, water, and vessel after mixing is 156°, what is the temperature of the furnace?

SOLUTION.—Substituting the values given in formula **27**,

$$156 = \frac{1 \times .1138 \times t_1 + 2 \times 75 + .5 \times .0951 \times 75}{1 \times .1138 + 2 + .5 \times .0951},$$

or $$156 = \frac{.1138\, t_1 + 153.56625}{2.16135},$$

or $$156 \times 2.16135 = .1138\, t_1 + 153.56625.$$

Hence, $$.1138\, t_1 = 183.60435,$$

or $$t_1 = \frac{183.60435}{.1138} = 1{,}613.4°. \quad \text{Ans.}$$

113. By means of formula **27**, the specific heat of a substance may be obtained. Thus, in

$$t = \frac{W_1 s_1 t_1 + W_2 s_2 t_2 + W_3 s_3 t_3 + \text{etc.}}{W_1 s_1 + W_2 s_2 + W_3 s_3 + \text{etc.}},$$

suppose that the specific heat s_3 is required and that all of the other quantities, including t, are known.

Then, solving the above equation for s_3,

$$t(W_1 s_1 + W_2 s_2 + \text{etc.}) + t W_3 s_3 = W_1 s_1 t_1 + W_2 s_2 t_2 + W_3 s_3 t_3 + \text{etc.},$$

or $$t W_3 s_3 - t_3 W_3 s_3 = W_1 s_1 t_1 - W_1 s_1 t + W_2 s_2 t_2 - W_2 s_2 t + \text{etc.},$$

or $$s_3 = \frac{W_1 s_1 (t_1 - t) + W_2 s_2 (t_2 - t) + \text{etc.}}{W_3 (t - t_3)}. \qquad (28.)$$

EXAMPLE.—A silver vessel weighing 13 ounces is suspended by a string; 1 pound 4 ounces of water having a temperature of 120° is poured into it, and in this is placed a piece of metal weighing 14 ounces and having a temperature of 100°. If the temperature of the vessel was 72°, and the temperature of the mixture is 117°, what is the specific heat of the piece of metal?

SOLUTION.—Using formula **28**, and letting W_1, s_1, and t_1 represent, respectively, the weight, specific heat, and temperature of the silver vessel; W_2, s_2, and t_2, the same for the water; and W_3, s_3, and t_3, the same for the piece of metal,

$$s_3 = \frac{W_1 s_1 (t_1 - t) + W_2 s_2 (t_2 - t)}{W_3 (t - t_3)}$$

$$= \frac{13 \times .057 (72 - 117) + 20 \times 1 (120 - 117)}{14 (117 - 100)} = \frac{-33.345 + 60}{238} = .112. \quad \text{Ans.}$$

All weights must be reduced to either pounds or ounces before substituting.

EXAMPLES FOR PRACTICE.

114. Solve the following:

1. How many units of heat are required to raise the temperature of 10 ounces of platinum from 80° to 2,000° ? Ans. 38.88 B. T. U.

2. In order to determine the specific heat of a certain alloy, a piece weighing 12½ ounces was heated to a temperature of 320°, and was then immersed in 2 pounds 6 ounces of water contained in a lead vessel weighing 4 pounds 7 ounces. The temperature of the water and of the vessel being 70°, what was the specific heat of the alloy if the temperature of the mixture was 79° ? Ans. .1202.

3. In order to determine the temperature of the interior of a chimney, a silver bar weighing 20 ounces is placed in it until it has attained the same temperature. It is then immersed in 1 pound of water contained in a brass vessel weighing 10 ounces. The temperature of the vessel and water being 65°, and of the mixture 98½°, what was the temperature within the chimney ? Ans. 596°.

4. (*a*) An iron casting weighing 3 tons is cooled from 2,100° to 100°; how many units of heat does it give up ? (*b*) If all this heat could be utilized, how many pounds of coal would it be equivalent to, assuming that 1 pound of coal gives out 14,500 B. T. U. during its combustion ? Ans. { (*a*) 1,557,600 B. T. U. (*b*) 107.42 lb.

LATENT HEAT.

115. In all that has been said in the preceding pages only the phenomena relating to sensible heat have been considered. If a quantity of pounded ice at a temperature of 32° be put in a vessel and held over the flame of a spirit lamp, heat passes rapidly into the ice and melts it; but a thermometer resting in this mixture of ice and water shows no tendency to rise; it will remain at 32° until all of the ice has been melted. Where has the heat gone that was supplied to the ice? This question was first investigated by Dr. Black, of Edinburgh, in 1760, and is easily explained by the modern dynamical theory of heat.

Dr. Black took a pound of water and a pound of ice, both having a temperature of 32°, and placed them in two vessels suspended in a chamber that was kept at as nearly uniform a temperature as possible. At the end of half an hour the temperature of the water was 39.2°, but the ice did not reach

that temperature till $10\frac{1}{2}$ hours had passed, being melted, of course, in the meantime. Dr. Black reasonably assumed that the ice received the same quantity of heat that the water did in each half hour, because it was placed in exactly the same position in regard to the surrounding air; that is to say, it received $39.2 - 32 = 7.2$ units of heat every half hour, or 14.4 units every hour, and $14.4 \times 10\frac{1}{2} = 151.2$ units in $10\frac{1}{2}$ hours; hence, it took $151.2 - 7.2 = 144$ units of heat to change the pound of ice at 32° into water at 32°. More accurate determinations have fixed this number as 142.65, and this value will be used in this section whenever the occasion arises for using it. If a pound of water having a temperature of 212° is mixed with a pound of water having a temperate of 32°, the temperature of the mixture will be $\frac{212 + 32}{2} = 122°$; the boiling water giving up 90° and the cold water receiving 90°, thus bringing both to a common temperature. If a pound of ice at a temperature of 32° is mixed with a pound of water at a temperature of 212°, the temperature of the mixture will be only 50.675°, instead of 122, as in the previous case. Here the water has given up $212 - 50.675 = 161.325$ units of heat in order to bring both bodies to a common temperature. Since the temperature of the ice was raised from 32° to 50.675°, it follows that $50.675 - 32 = 18.675$ units of heat were used to raise the temperature of the ice *after it had been melted* into water, and $161.325 - 18.675 = 142.65$ units of heat were necessary to convert the ice at 32° into water of the same temperature. This extra number of units of heat, which is necessary to convert a solid into a liquid of the same temperature without raising the temperature of the solid, is called the *latent heat of fusion*, and the temperature at which this change of state in the body takes place is called the *melting point*, or *temperature of fusion.* All solids probably have a latent heat of fusion, the word *probably* being used because some solids have never been melted, except at such high temperatures that accurate measurements are not possible, but its value varies greatly for different substances, being greater

for ice than for any other known solid, while for frozen mercury its value is only 5.09; that is, to change 1 pound of frozen mercury at its temperature of fusion (−37.8° F.) into liquid mercury of the same temperature requires only 5.09 units of heat. Now, it is reasonable to suppose that if it requires 142.65 units of heat to convert a pound of ice at 32° into water at 32°, then the same number of heat units would be given up when water at 32° is changed into ice at 32°; experiment has shown that this is true.

116. If water is heated to its boiling point of 212° under a constant pressure of 14.69 pounds per square inch, it has been found by experiment that it will require about 966 units of heat per pound of water to change it into steam at 212°. This extra number of units of heat necessary to convert a liquid into a gas, or rather vapor, of the same temperature and pressure, is called the *latent heat of vaporization*, and the temperature at which this change of state takes place is called the *temperature of vaporization.*

117. According to the modern theory of heat, the extra quantity of heat necessary for a change of state of a body is used in forcing the molecules of a body farther apart, and in overcoming the force of cohesion. This latent heat is not lost, but performs work in giving additional energy to the molecules of a body, and it always reappears when the body resumes its former state. Thus, for instance, a pound of steam under a pressure of 1 atmosphere contains $966 + 180 = 1{,}146$ units of heat more than a pound of water at 32°. Hence, if 1 pound of steam at 212° is mixed with $\frac{966}{180} = 5.37$ pounds of water at 32°, the temperature of the mixture will be exactly 212°, or the boiling point of water; in other words, the steam raised 5.37 pounds of water from the freezing point to the boiling point without lowering its own temperature, by merely changing from steam into water. If a pound of water at a temperature of 32° is changed into ice of the same temperature, 142.65 units of heat will be given up during this change of state.

118. In the following table are given the temperatures of fusion and of vaporization, and the latent heats of fusion and vaporization whenever they have been determined with sufficient accuracy:

TABLE 6.

Substance.	Temperature of Fusion.	Temperature of Vaporization.	Latent Heat of Fusion.	Latent Heat of Vaporization.
Water	32°	212°	142.65	966.6
Mercury	−37.8°	662°	5.09	157
Sulphur	228.3°	824°	13.26	
Tin	446°		25.65	
Lead	626°		9.67	
Zinc	680°	1,900°	50.63	493
Alcohol	Unknown	173°		372
Oil of turpentine	14°	313°		124
Linseed oil		600°		
Aluminum	1,400°			
Copper	2,100°			
Cast iron	2,192°	3,300°		
Wrought iron	2,912°	5,000°		
Steel	2,520°			
Platinum	3,632°			
Iridium	4,892°			

119. The following examples will show the purpose of Tables 5 and 6:

EXAMPLE 1.—How many units of heat are required to change 12 pounds of ice at a temperature of −20° C. into steam at 212° F.?

SOLUTION.—By formula **19**, $t_f = (\frac{9}{5} \times -20) + 32 = -4°$ F. This is equivalent to $32° + 4° = 36°$ F. below the freezing point. In Table 5, the specific heat of ice was given as .504; hence, it will take $12 \times 36 \times .504 = 217.728$ B. T. U. to raise the temperature of 12 pounds of ice from −4° to 32°. To convert this ice into water of 32° will require $142.65 \times 12 = 1{,}711.8$ B. T. U. To raise this water from 32° to a temperature of 212° will require $12 \times 180 = 2{,}160$ B. T. U. To convert it into steam of 212° will require $966.6 \times 12 = 11{,}599.2$ B. T. U. The total number of units of heat required will be $217.728 + 1{,}711.8 + 2{,}160 + 11{,}599.2 = 15{,}688.728$ B. T. U. Ans.

EXAMPLE 2.—How many units of heat will it take to evaporate 25 pounds of mercury from a temperature of 70°?

SOLUTION.—The temperature of vaporization of mercury is 662°, and the specific heat is .0333; the increase in temperature from 70° will be $662° - 70° = 592°$. The number of units of heat required will be $25 \times 592 \times .0333 = 492.84$ heat units. The latent heat of vaporization is 157; hence, $492.84 + 25 \times 157 = 4{,}417.84$ B.T.U will be required. Ans.

120. A solid may be changed into a liquid, not only by melting it, but also by dissolving it, as salt or sugar is dissolved in water. Since the particles of the solid body must be torn asunder in opposition to the forces that hold them together, it is reasonable to suppose that a certain amount of heat will be required to do this. That such is a fact may be easily proved by any one having a thermometer. Place a thermometer in a vessel of water and leave it there until it indicates the temperature of the water; then put in the water some salt or sugar and stir so as to make it dissolve more quickly, and it will be found that the mercury has fallen several degrees. In fact, if any solid is dissolved in a liquid that does not act chemically upon it, the temperature of the mixture will be lower than if the solid did not dissolve. It is this principle that is taken advantage of in the so called freezing mixtures. A mixture of 1 part of nitrate of ammonia and 1 part of water will reduce the temperature from 50° to 4°, a fall of 46°. The effects are still more striking when both bodies are solids, one of which is already at the freezing point. Thus, a mixture of 2 parts of snow or finely powdered ice and 1 part of common salt will reduce the temperature from 50° to 0°, a range of 50°; while a mixture of 4 parts of potash and 3 parts of snow or powdered ice will lower the temperature from 32° to −51°, a range of 83°.

121. Latent heat plays an important part in every-day life. It takes a long time and severe cold to freeze the water of a river to any depth, even though the thermometer goes far below the freezing point. This is because 142.65 units of heat must be given up by every pound of water, after being brought to the freezing point, before the ice can form.

If it were not for this, the rivers, lakes, and other bodies of water would be frozen solid as soon as the water reached the freezing point and would be melted as soon as the temperature went above that point. In the spring, all of the snow on the hills would be melted during a warm day, and great floods would be the consequence. As it is, 142.65 units of heat must be supplied to every pound of snow at 32° to convert it into water at 32°.

EXAMPLES FOR PRACTICE.

122. Solve the following:

1. If a pound of steam at 212° and 7 pounds of ice at 32° are mixed, what will be the resulting temperature? Ans. 50.5°.

2. How many units of heat are required to vaporize 10 pounds of mercury from a temperature of 100°? Ans. 1,757.146 B. T. U.

3. How many pounds of oil of turpentine at 60° can be vaporized by 1 pound of coal, if the coal gives out 14,500 B. T. U. during combustion? Ans. 62.56 lb.

4. How many pounds of water at 32° can be vaporized by 1 pound of coal? Ans. 12.646 lb.

5. How many pounds of coal are required to raise 100 pounds of wrought iron from 85° to its melting point? Ans. 2.219 lb.

SOURCES OF HEAT.

123. Heat is derived from the following sources:

1. *Physical sources*, which include the radiation of heat from the sun, terrestrial heat, change of state in bodies, and electricity.

2. *Chemical sources*, resulting from chemical combinations, such as combustion, oxidation, etc.

3. *Mechanical sources*, comprising friction, percussion, and pressure.

124. Physical Sources.—1. The most intense of all of the sources of heat is the sun. The majority of scientists are of the opinion that all of the heat received or given up

by the earth has, or has had, its source in the sun. It would be out of place here to enter into this theory fully. It is the amount of heat radiated from the sun and received by the earth that causes the change of seasons, that causes the waters in the rivers, lakes, and seas to evaporate and form the clouds, to be again precipitated as rain or snow. Without it, no living thing, animal or vegetable, could exist.

2. The earth possesses a heat peculiar to itself, called *terrestrial heat.* When a descent is made below the surface, the temperature is found to gradually increase. This is not caused by the heat radiated from the sun, for the material comprising the earth is such a poor conductor that the heat of the sun's rays penetrates only a very short distance below the surface. The explanation usually given for this phenomenon is that the interior of the earth is in a molten condition. The terrestrial heat exerts but a slight effect, not raising the temperature of the surface more than $\frac{1}{20}$ of a degree.

3. If a liquid is poured upon a finely divided solid, as a sponge, flour, starch, roots, etc., the temperature will be increased from 1° to 10°, according to conditions. This phenomenon may be called *heat produced by capillarity.*

4. The heat produced by a change of state has already been described; it is the heat given off when a body is converted from a gas or liquid to a liquid or solid.

5. Extremely high temperatures may be produced by the electric current. By means of it, quicklime, firebrick, osmium, porcelain, and several other substances, which until very recently have resisted every attempt to melt them, may be made to run like water.

125. Chemical Sources.—Whenever two or more substances that act chemically upon one another are brought together and allowed to combine, heat is evolved; and whenever the heat caused by this chemical union is sufficiently intense to raise the resultant substances to a temperature at which they emit light, this act of union is termed *combustion.*

This subject will not be treated here, but will be considered fully by itself in *Theoretical Chemistry.*

126. Mechanical Sources.—1. The friction between any two bodies rubbed together produces heat. Rubbing one hand briskly against the other will soon make the hands too warm for comfort. The friction between a journal and its bearing causes heat; the heat causes the journal and bearing to expand, the journal expanding more rapidly on account of being smaller and being heated more quickly; the expansion causes a greater pressure on the bearing, producing more friction and heat. If the bearing is not properly oiled, the heat will become so intense in a short time that the soft metal in it will melt. When shooting stars strike the earth's atmosphere, their velocity is so great (sometimes as high as 150 miles a second) that the friction of the atmosphere causes them to take fire almost instantly. Wherever there is friction, there is heat.

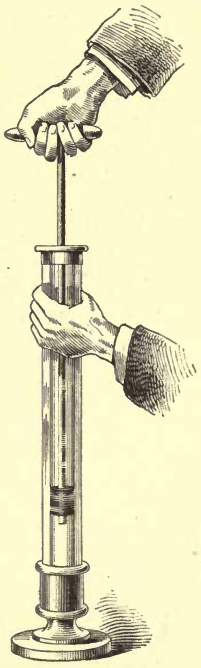

FIG. 17.

2. Heat is also generated by percussion. The repeated blows of a hammer upon a piece of iron, lead, or other metal will soon make it quite hot.

3. The generation of heat by pressure was spoken of in connection with gases; that is, the temperature rises when a gas is compressed. This is also true of solids and liquids, but the results are not so marked in their cases. The production of heat by the compression of gases is easily shown by means of the *pneumatic syringe* shown in Fig. 17. This consists of a glass tube with thick sides, hermetically closed with a leather piston. At the bottom is a small cavity in which a piece of cotton, moistened with ether or carbon disulphide, is placed. The tube being filled with air, the piston is suddenly

plunged downwards. Thus compressed, the air generates so much heat that the cotton is ignited, and can be seen burning when the piston is suddenly withdrawn. The ignition of the cotton in this experiment indicates a temperature of at least 570°, since it will not ignite at a lower temperature.

LIGHT.

PROPAGATION OF LIGHT.

127. When the nerves of the retina of the eye are stimulated, the sensation is commonly described by saying that *light* is seen. Light is, then, a pure sensation; and it may be caused in several ways. Ordinarily it is produced as the result of some action that is taking place at some distance from the observer—such as the combustion of gas in a lamp flame, or the chemical processes of the sun.

128. Theory of Light.—Sir Isaac Newton assumed that luminous bodies emit infinitely small particles in straight lines, which, by penetrating the transparent parts of the eye and falling upon the nervous tissue, produce light. Modern philosophers, however, in developing the theory of sound, came to the conclusion that light might be the effect of *undulations*, or *little waves*, propagated with inconceivable velocity through the highly elastic medium of excessive tenuity called *ether*, which fills all space and lies between the particles of material substances.

129. Transparent, Translucent, and Opaque Bodies. Bodies that allow light to pass, so that objects can be clearly distinguished through them, are called *transparent;* the air, water, and clear glass are common examples. Oiled paper, roughened glass, and porcelain allow light to pass, but objects cannot be clearly distinguished through them; they are *translucent* bodies. *Opaque* bodies, such as wood or stone, do not

allow light to pass through them. Opaque bodies, when very thin, are translucent, and transparent bodies, when very thick, become translucent and even opaque.

130. Propagation of Light.—A **medium** is a substance, as air and glass, through which light passes. If the density and composition of the medium is the same in all parts, it is called *homogeneous;* water, carefully prepared glass, and the atmosphere, if we consider only a small portion of it, are homogeneous mediums. In a homogeneous medium light is propagated in straight lines; this is our every-day experience. The rays of the sun, as they trace their way through our room, are straight; in order that light may pass through a tube, the tube must be straight.

131. Ray and Pencil of Light.—A light proceeding from a luminous body is supposed to be made up of straight lines, called *rays of light.* A number of rays form a *pencil of light.*

132. Velocity of Light.—A ray of light emitted from a luminous body proceeds in a straight line with extreme velocity. Through a series of astronomical observations, an approximate knowledge of this velocity has been obtained. As the moon revolves about the earth, so revolve the satellites of Jupiter about that planet, and the time of revolution of each satellite has been ascertained from its periodical entry into or exit from the shadow of the planet. The time required by one satellite is only 42 hours. The Danish astronomer Römer observed that this period appeared longer when the earth in its passage around the sun increased the distance between Jupiter and itself, and again that the periodic time became shorter when the earth moved towards Jupiter. Although the difference is rather small for a single revolution of the satellite, it increases by the addition of many revolutions during the passage of the earth from its nearest to its greatest distance from Jupiter, a passage that occupies about 6 months, till it amounts to 16 minutes and 16 seconds. From this, Römer concluded that the light of

the sun reflected from the satellite required that time to pass through a distance equal to the diameter of the orbit of the earth, and, since this space is, approximately, 190,000,000 miles, the velocity of light must be about 190,000 miles per second.

REFLECTION AND REFRACTION.

133. Let it be assumed that a ray of light is deflected by a mirror M, Fig. 18, so that it strikes point N of a horizontal mirror L, and is from there reflected in the direction

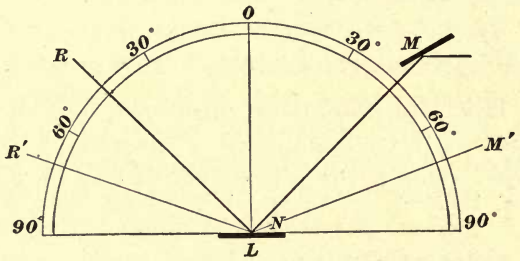

FIG. 18.

R. If we then draw a line from point N to O at right angles to the mirror and measure the angles MNO and ONR, we find that these two angles are equal, and we further notice that the two rays lie in a plane which is at right angles to the mirror N. The line NO perpendicular to the mirror is called the *normal*. The ray MN is called the *incident ray* and the ray NR is the *reflected ray*. The angle that the incident ray makes with the normal is the *angle of incidence*, and the angle made by the normal and the reflected ray is the *angle of reflection*.

134. The general laws of reflection are as follows:

1. *The angle of incidence is equal to the angle of reflection.*

2. *In whatever position the incident ray meets the reflecting surface, a plane can always be so placed that it passes through the incident ray, the reflected ray, and the normal.*

135. The same laws also hold good if a mirror or any other reflecting surface be curved, as a portion of a sphere, since a curve may be considered as made up of a multitude of minute planes. Parallel rays, however, cease to be parallel when reflected from such a surface; they become convergent or divergent according as the reflecting surface is concave or convex.

Suppose that $A\ B$, Fig. 19, is a section of a concave mirror. Point C, the center of the sphere, is known as the center of

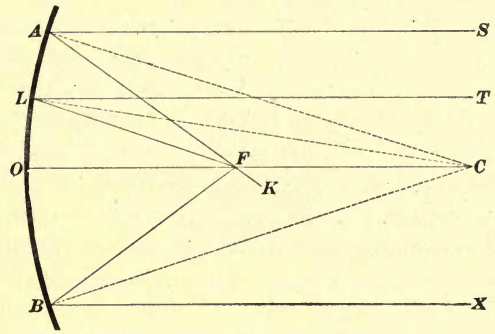

FIG. 19.

curvature, point O as the apex, and $C\ O$ as the principal axis. A very small portion of the mirror at any point A will be a plane, of which $A\ C$, the radius, will be the normal; and if any incident ray, as $S\ A$, strikes the mirror at A, the reflected ray will be $A\ F$ (angle $S\ A\ C$ = angle $F\ A\ C$); the same is true for any small portion of the mirror, as, for instance, L, O, and B. If the arc $A\ B$, the section of the reflecting surface, is small compared to the radius $C\ O$, it is found, both by measurements and experiments, that all incident rays, as $S\ A$, $T\ L$, $X\ B$, parallel to the principal axis, after reflection, pass through the same point F in the principal axis. This point, which is midway between C and the mirror, is called the *principal focus*, and the distance between F and O is known as the *focal length*.

136. In Art. **130** it was stated that the rays of light proceed in straight lines; this, however, is true only as long

as the medium through which these rays pass is homogeneous. Should, however, the density or chemical nature of the medium through which the ray travels vary, the ray of light is bent from its original course into a new one; or it is said to be *refracted*. Fig. 20 shows a thick plate of glass through which a ray of light is supposed to travel, *B* being the point of the surface where the ray *A* enters. This ray, instead of proceeding onwards in the direction *C*, will be bent downwards towards point *D*, and on leaving the plate and entering into the air on the other side, will again be bent, but in an opposite direction, so as to travel parallel to the continuation of the original path, provided there is the same medium on the upper as on the lower side of the plate. The general law of refraction may, then, be expressed as follows: *When a ray of light passes from a rarer to a denser medium it is usually refracted towards a line perpendicular to the surface of the latter, and, conversely, when it leaves a dense medium for a rarer one, it is refracted away from a line perpendicular to the surface of the denser substance.* In the former case the angle of incidence is greater than that of refraction; in the latter it is less. In Fig. 20, angle *A B E* is the *angle of incidence*, and *D B E'* is the *angle of refraction.*

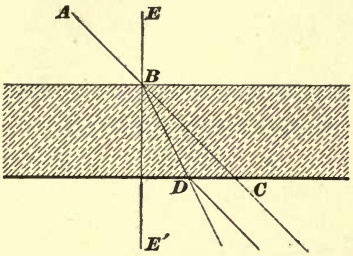

FIG. 20.

137. The amount of refraction, for the same medium, varies with the obliquity with which the ray strikes the surface. When perpendicular to the latter, the ray passes without change of direction; and in other positions the refraction increases with the obliquity.

138. Let *A*, Fig. 21, represent a ray of light falling upon the surface of a heavy plate glass at the point *B*. From this point let a perpendicular fall and be continued into the new medium, and around *B* as a center let a circle be drawn. Then,

according to the general law of refraction, stated in Art. **136,** the refraction must be towards the perpendicular, in the direction *B C*, for example. If now the lines *a a* and *a' a'*, which are the *sines* of the angles of incidence and refraction, respectively, are drawn at right angles to the perpendiculars, and their lengths compared by means of a scale, these lengths will be found, in this case, to be in the ratio of 3 : 2. Or, if another ray such as *E* is taken, which is refracted in the same manner towards *F*, the bending being naturally greater from the increased obliquity of the ray, and if the sines of the new angles of incidence and refraction are then compared, they will be found to bear to each other the same ratio; i. e., 3 : 2. This fact may be, then, expressed by the rule : *So long as the light passes from one to the other of two media, the ratio of the sines of the angles of incidence and refraction remains constant.*

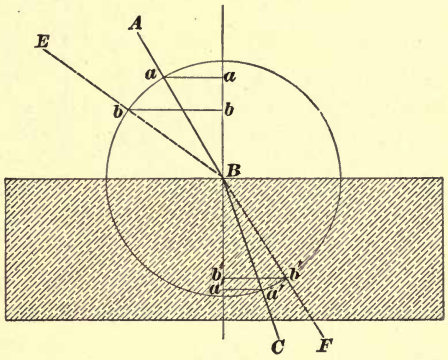

FIG. 21.

139. Different bodies possess different refractive power, and, as a rule, the densest substances refract most. The method adopted for describing the comparative refractive power of different bodies is to state the ratio of the sine of the angle of incidence in the first medium to the sine of the angle of refraction in the second medium; this ratio is known as the *index of refraction* of the two substances. It is greater or less than unity, according as the second medium is denser or rarer than the first.

If the index of refraction of any particular substance is once known, the effect of the latter upon a ray of light entering it at any angle can be easily calculated by the law of sines.

140. In Table 7 the indices of refraction of several substances are given. It is assumed that the ray of light passes into the substances in question from the air.

TABLE 7.

INDICES OF REFRACTION.

Ice	1.30	Carbon bisulphide	1.70
Water	1.34	Garnet	1.80
Fluorspar	1.40	Phosphorus	2.20
Plate glass	1.50	Diamond	2.50
Rock crystal	1.60	Lead chromate	3.00
Chrysolite	1.69	Cinnabar	3.20

141. Some Results of Refraction.—A stick partly submerged in water appears to be bent at the surface; the tip of the stick and all that is under water seems to be raised. The atmosphere is less dense the higher we ascend, and we can assume it to be made up of layers of air of various densities. The rays of light from the sun or a star, instead of descending in straight lines, are refracted at every layer. Thus, the light from S, in Fig. 22, reaches the eye as if it came from S'; that is, a star appears higher in the heavens than it really is. For the same reason, the sun is seen before it is above, and after it is below, the horizon.

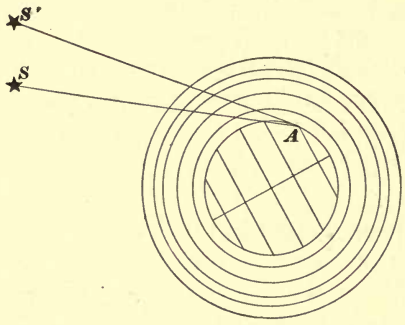

FIG. 22.

142. Irregular Refraction.—The rays of light are straight in air and straight in water, provided the density is uniform. This is not the case if from any cause the density is not uniform. The quivering of objects, when seen over heated coke or on a hot day, is due to unequal refraction, the density being constantly changing.

143. Total Reflection.—When a ray of light passes from one medium into another that is less refracting, as from water into air, it has been seen in Art. **136** that the emerging ray is again bent from its course. Hence, when light is propagated in a mass of water from C to A, Fig. 23, there is always a value of the angle of incidence $C\,A\,B$ such that the angle of refraction $B\,A\,D$ is a right angle, in which case the refracted ray emerges parallel to the surface of the water. This angle $C\,A\,B$ is called the *critical angle*, since, for any greater—$E\,A\,B$, for instance—the incident ray can no longer emerge, but undergoes reflection. This is called *total reflection*, because the incident light is entirely reflected. The critical angle for water and air is $48\frac{1}{2}°$.

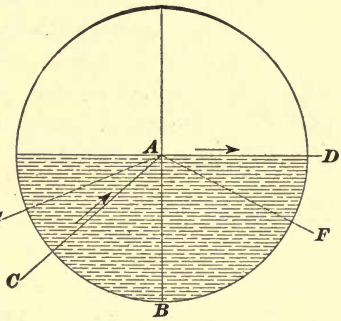

FIG. 23.

If you raise a glass of water above your head you can easily find a position in which the rays from the eye to every part of the under surface make a greater angle than $48\frac{1}{2}°$ with the normal. A brilliant mirror due to the total reflection of light will be observed.

144. Transparent glass when finely ground loses its transparency; light is diffused at the surface, and the few rays that pass through the upper layer of the fine particles are refracted in passing from the glass to the air between the surfaces; the result of this combined irregular reflection and refraction is that the glass appears opaque.

If a liquid, whose refractive index is the same as that of glass, is poured over the glass, the result will be the same, in respect to the action of light, as if the surface were made smooth again and all the spaces were filled with glass; that is, powdered glass becomes in its effects a glass plate and is again transparent.

145. Prisms.—A prism may be defined as a portion of a transparent medium bounded by plane inclined surfaces. The line along which these surfaces meet is called the *refracting edge*, and the angle between them, the *refracting angle* of the prism. The side opposite the refracting edge is called the *base* of the prism, and any section through it, perpendicular to this edge, is called a *principal section.* If a ray of light falls upon one of the inclined surfaces of a prism, the change in the direction of the ray—that is, its deviation—instead of being neutralized at the second surface, is still further increased.

Let ray A, Fig. 24, be incident to the surface PQ of a prism at O. At O the refraction will be towards the normal

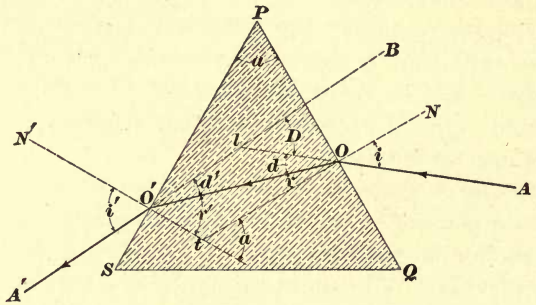

FIG. 24.

and the ray will be refracted to O'. Draw normals at O and O'. At O' the second refraction takes place, but since the face PS is inclined to the face PQ, the second deviation is in the same direction as the first; that is, towards the base of the prism. Evidently the deviation of the ray at the first surface is $A\,O\,N - t\,O\,O'$ or $d = i - r$, and at the second is $A'O'N' - t\,O'O$ or $d' = i' - r'$. The total deviation is the sum of the partial deviations, or $D = d + d' = (i - r) + (i' - r')$. That is, the total deviation in the figure, or the angle $A\,l\,B$, or D, between the incident and emergent rays is equal to the sum of the partial deviations $d + d'$, or $l\,O\,O'$ and $l\,O'\,O$, since the exterior angle of the triangle is equal to the sum of the two interior angles. Moreover, the deviation may be

obtained in terms of the index (n) and the prism angle. Since $D = (i-r)+(i'-r')$ or $(i+i')-(r+r')$, and as the prism angle is equal to $r+r'$, the deviation $D = (i+i')-a$. If the prism be thin and the incident ray nearly normal, we may write $i = nr$ and $i' = nr'$, whence $i+i' = n(r+r')$ or na, and $D = na-a$ or $(n-1)a$; which means that the deviation is equal to the prism angle multiplied by the index of refraction less 1.

146. When the deviation produced by a prism is a minimum, the angles formed by the ray with the normals within the prism are equal, as are also the angles without it; that is, $r = r'$ and $i = i'$. Consequently, we can simplify the formula for deviation, $D = (i+i')-(r+r')$, by writing it $D = 2i-2r$. In Fig. 25 we see that the angle at t formed by the two normals is for that reason equal to a, the refractive angle of the prism. But, being exterior to the triangle $O\,O'\,t$, it is equal to the sum of the two interior angles, that is, $r+r'$ or $2r$; therefore, $a = 2r$ and $r = \frac{1}{2}a$. Hence, $D = 2i-a$; and $i = \frac{1}{2}(D+a)$. By the formula for the index, $n = \dfrac{\sin i}{\sin r}$, we have $n = \dfrac{\sin \frac{1}{2}(D+a)}{\sin \frac{1}{2}a}$.

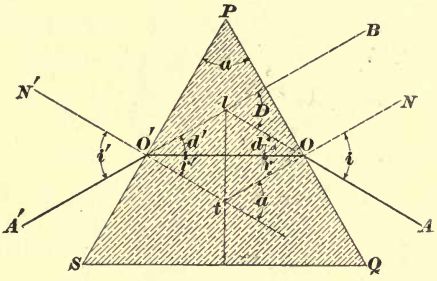

FIG. 25.

To determine the refractive index of a substance, therefore, it is only necessary to measure (1) the angle a of the prism and (2) the deviation angle D when this angle has its minimum value.

147. **Lenses.**—A lens is any portion of a transparent medium bounded by curved surfaces, and may be considered as being made up of a large number of prisms. Suppose we have a number of small prisms and we place the prism B,

Fig. 26, with the largest angle so that a ray of light A is refracted to C; then, evidently, if a prism H, exactly like

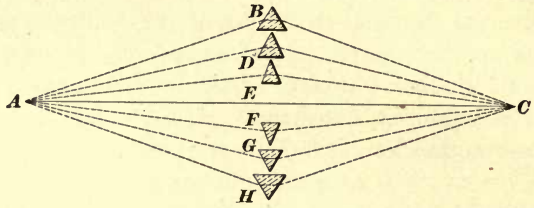

FIG. 26.

B, be placed as in the figure, it also will refract light to C. By using D and G with smaller angles (less deviation), we can find, by trial or calculation, positions at which they also refract rays from A to C; we similarly use E and F, and between E and F we might place a plate of glass with parallel sides. Thus, a number of rays from A can be made to converge to C. The arrangement shown at (b), Fig. 27, will have exactly the same effect as (a), Fig. 27, if the surfaces of A, B, C, etc. in (b) are inclined at the same angles as the surfaces of A, B, C, etc. in (a). A very large number of such prisms fitted together as in (b), Fig. 27, will form a body whose section is as represented at (c), and the surface will be practically spherical.

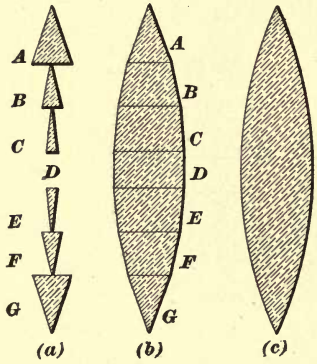

FIG. 27.

It has been found by experiments and calculations that lenses whose surfaces are parts of the surface of a sphere act like the prisms in (a), Fig. 27, if the surfaces are small compared with the whole surface of the sphere.

148. The names given to the lenses in Fig. 28 are: A, *double convex*—both surfaces convex; B, *plano-convex*—one surface convex, one plane; C, *concavo-convex converging*—

one surface convex, one concave; *D*, *double concave*—both surfaces concave; *E*, *plano-concave*—one concave, one plane;

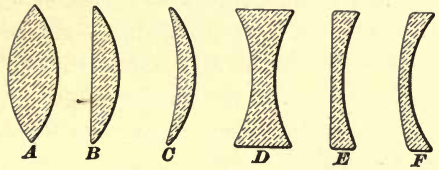

FIG. 28.

F, *concavo-convex diverging*—one concave, one convex. *E* and *F* are also known as *meniscus lenses*.

149. Surfaces of lenses are usually parts of the surfaces of spheres of equal radii. Fig. 29 shows how a double-convex lens is formed. The line *A B* passing through the

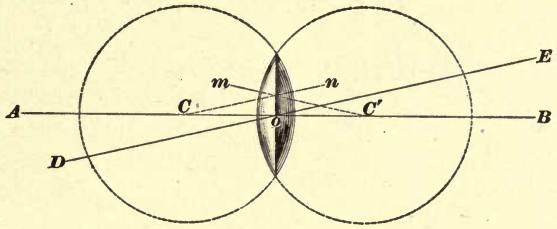

FIG. 29.

center is the principal axis; *C n* and *C' m* are the normals at *n* and *m*, and *C*, *C'* are the centers of curvature. The point *o* is called the optical center. The point where the rays parallel to the principal axis come to a focus, after refraction, is called the principal focus. (See Art. **135.**)

THE SPECTRUM.

150. Dispersion.—It is an every-day experience in experiments with prisms and lenses to observe that the images are tinged with colors, reminding us of the hues of the rainbow. These colors are due to refraction. If a ray of light is admitted into a dark room through a small hole in a shutter, or otherwise, and is allowed to fall upon a glass

prism, and if a screen is put in the path of the rays, as shown in Fig. 30, there is refraction; but, instead of an image, we find on the screen a band of color, called a *spectrum*. This separation of light into its component parts is called *dispersion*. The light in passing through the prism has been decomposed and we can distinguish in order red (the least refracted of the rays), orange, yellow, green, blue, indigo, and violet (the most refracted).

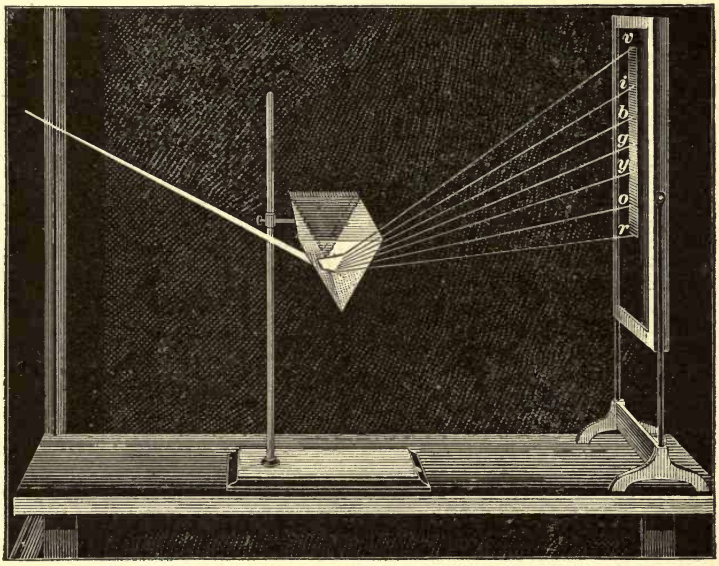

FIG. 30.

Sir Isaac Newton was the first to investigate this phenomenon, and drew from it the inference that white light is composed of these seven primitive colors, the rays of which are differently refrangible by the same medium and are thus capable of being separated.

151. Bodies of the same refractive power, of course, do not equally disperse the differently colored rays to the same extent; if, for instance, the principal yellow or red rays are equally refracted by two prisms of different substances, it

does *not* necessarily follow that the blue or violet rays are also similarly affected. Hence, prisms of different varieties of glass or other transparent substances give, under similar circumstances, very different spectra, both in regard to the length of the image and the relative extent of the colored band.

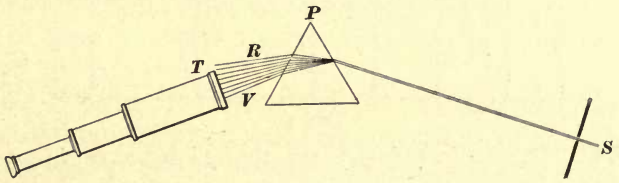

FIG. 31.

152. As the appearance of the spectrum also varies with the nature of the light, an apparatus called a **spectroscope** has been invented to investigate these variations. Fig. 31 illustrates the principle of this instrument. The apparatus is so arranged that the light after passing through a fine slit *S* impinges upon a flint-glass prism *P*, by which it is dispersed. The decomposed light emerges from the prism in several directions between *R* (red rays) and *V* (violet rays); and the spectrum thus produced is observed by the telescope *T*. Only a part of the spectrum enters the telescope at one time, but the several parts may be readily examined by slightly turning either the telescope or the prism.

153. If sunlight is allowed to fall upon the slit of the spectroscope, numerous fine black lines, of different degrees of breadth and shade, are observed. These lines are always present, and always occupy exactly the same relative position in the solar spectrum. They indicate the absence in sunlight of particular rays, and they may be considered as shadows or spaces where there is no light. They were discovered in 1802 by Dr. Wollaston, and subsequently were more thoroughly investigated by Fraunhofer, a German optician. They are generally known as *Fraunhofer's lines*. Some of them, in consequence of their peculiar strength and relative

position, may always be easily recognized; the more conspicuous are shown in Fig. 32.

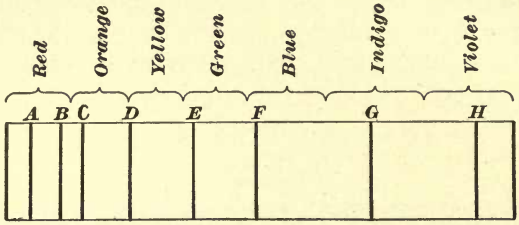

FIG. 32.

154. In the last few years the existence of these lines has become a matter of very great importance, as through them the determination of the chemical constitution of the sun and fixed stars has become possible. The spectra of the moon and planets, which are indirectly produced by the reflected light of the sun, exhibit, of course, the same lines in unaltered position, while in the spectra of the fixed stars, though dark lines also occur, these stellar lines are entirely different from those seen in direct or reflected sunlight. Hence, the conclusion has been drawn that the Fraunhofer's lines are in some way produced in the body of the sun itself, but it is only recently that the cause of their production has been discovered by Kirchhof and Bunsen, and thus the foundation laid for the science of *solar* and *stellar chemistry*.

155. It has long been known that certain chemical substances, especially the salts of the alkalies and alkaline earths, when strongly heated in the blowpipe flame, or other nearly colorless flames, impart to that flame a peculiar color, by the occurrence of which the presence of these substances can be detected. If these colored flames are examined by means of a spectroscope, it is at once seen that the light thus refracted differs essentially from white light, because it consists only of a particular set of rays, each flame giving a spectrum containing a few *bright bands*. The spectrum of the yellow soda flame, for instance, contains only one fine bright yellow line, while the purple potash flame exhibits a spectrum in which two bright lines appear, one lying at the extreme red

end and the other at the extreme violet end. These peculiar lines are always produced by the same chemical element, and by no other known substance, and the position of these lines always remains unaltered. When the spectrum of a flame tinted by a mixture of sodium and potassium salts is examined, the yellow ray of sodium is found to be confined to its own position, while the potassium red and purple lines are just as plainly visible as they would have been had no sodium been present.

Each of the colored flames exhibited by the salts of strontium, barium, lithium, and calcium likewise produces a peculiar spectrum. By means of these spectra the presence or absence of very minute quantities of these substances when mixed together can be ascertained by simply observing the presence or absence of the peculiar bright bands characteristic of the particular body.

156. If the positions of the dark lines in the solar spectrum are carefully compared in a powerful spectroscope with those of the bright lines in the spectra of certain metals, such, for instance, as iron, magnesium, and sodium, it is seen that each of the bright lines of the particular metal coincides not only in position, but also in width, with a dark solar line; so that if a spectroscope is so arranged that a solar and metallic spectrum is suffered to fall one below the other in the field of the telescope, the bright lines of the metal *all* appear as continuations of the dark solar lines. In the case of iron alone more than sixty such coincidences have been observed. With other metals, such as gold, antimony, lithium, etc., no single coincidence has been noticed, while all the lines of certain other metals have their dark representatives in the sun's spectra.

These coincidences could not be a mere chance. They gave rise, therefore, to the hypothesis that certain dark lines in the spectra of the sun, which are coincident with the bright lines in the spectra of iron, indicate the presence of iron in the sun. This hypothesis was strengthened through experiments of Foucault in 1849, who had reversed the yellow

double line in the spectrum of the electric arc into a double dark one. Kirchhof further investigated this subject and found that in like manner the spectra of many substances could be reversed; thus, the bright yellow soda lines, coincident with Fraunhofer's line *D* (see Fig. 32), can be made to appear as a dark line by allowing the rays of a strong white line to pass through a flame colored by soda, and then to fall upon the slit of a spectroscope. Instead of then seeing the usual soda spectrum, consisting of a bright yellow double line upon a dark ground, a double dark line identical in position and width with the soda line will appear to intersect the spectrum of the white light.

This fact, being the missing connecting link, at once thoroughly established the theory that by observing the coincidences of these dark lines with the bright lines of our metals, we are able to arrive at a knowledge of the occurrence of such metals in the solar atmosphere.

157. The attention of the student may here be drawn to the fact that light and heat are reflected and refracted similarly, mirrors and lenses having practically the same focus for each. Between the smallest particles of matter there is supposed to exist a medium called ether; it is by the vibration of the ether spheres that light and radiant heat are transmitted. The waves are of different lengths; that is, the periods of vibration of the particles differ. The longest rays cause the sensation of heat, next come the rays that cause the sensation of red color, and shorter than these are the rays that cause the sensation of orange color. As the rays become shorter the other colors of the spectrum are produced, the shortest of the light waves giving the sensation of violet color. There are yet still shorter rays than the violet, known as *actinic rays;* they do not affect the eye as regards color, but are, however, important in photography, from their power to affect certain chemical substances.

When a body is heated it possesses energy; this energy sets the ether spheres in motion, and the first waves transmitted are the heat waves. As the temperature of the body

rises, the increased energy starts other waves. First we have those that cause the sensation of red color; the body is at red heat. As the temperature further rises shorter waves are started, and the color changes until, when all the light waves are in motion, the composite effect is white light; the body is at white heat, as we commonly express it.

Ice transmits the light waves, but absorbs the heat waves, whose energy melts the ice. A solution of iodine in carbon disulphide absorbs the light waves—that is, no light passes—but transmits the heat waves. Red glass allows the long waves to pass, but absorbs the short waves.

A simple color, then, depends upon the wave length, that is, upon the period of vibration; the intensity of such a color depends upon the amplitude of the vibrations. The wave lengths are, naturally, exceedingly small, there being 33,000 waves per inch in a red ray, and 64,000 per inch in a violet ray.

DOUBLE REFRACTION AND POLARIZATION.

158. Double Refraction.—A large number of crystals possess the property of double refraction, which means that if a ray of light passes through such a crystal, it becomes split or divided into two rays; one, which is called the *ordinary ray*, follows the general law of refraction, while the other, which is known as the *extraordinary ray*, takes an entirely different, new course, which depends on the position of the crystal. This remarkable property is found in nearly all bodies that crystallize in any other than the cubical form. The phenomenon is explained by assuming that the *ether* in these doubly refracting bodies is not equally elastic in all directions, and that, consequently, the vibrations, in certain directions at right angles to each other, are transmitted with unequal velocity. A transparent variety of calcite—found in Iceland, and hence known as *Iceland spar*—possesses this property in a marked degree. If a piece of this substance is placed upon a sheet of white paper on which an ink spot or any other mark has previously been made, two images of the mark will be seen.

159. Polarization.—If a ray of light $C D$ is thrown upon a plate of glass $A A$, as shown in Fig. 33, at an angle of 56° 45′, that portion of the ray which is *reflected* will be found to have acquired properties it did not before possess; for, on throwing it at the same angle upon a second glass plate $B B$, it will be observed that there are two particular positions of the latter (indicated by $B' B'$); namely, those in

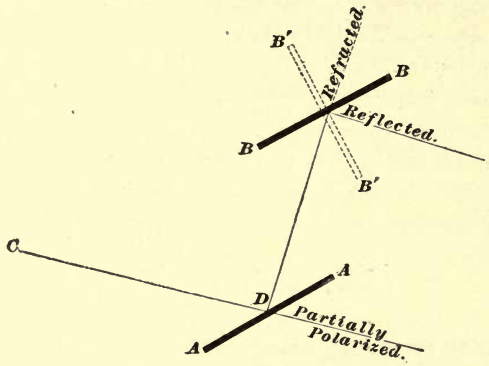

FIG. 33.

which the planes of incidence are at right angles to each other, when the ray of light is no longer reflected, but entirely refracted. Light that has undergone this change is said to be *polarized.*

The other portion of the ray that passes *through* the first, or polarizing, plate $A A$ is also to a certain extent in this peculiar condition—i.e., polarized—and by using a number of similar plates, placing one behind the other, this effect may be greatly increased. It must, however, be remarked that the light polarized by transmission is in an opposite state to that polarized by reflection; that is, when examined by the second, or analyzing, plate, held at the angle before mentioned, it will be seen to be reflected when the other is transmitted, and to be dispersed when the first is reflected.

160. Angle of Polarization.—Not every transparent substance is capable of polarizing light. Glass, water, and a number of other substances, however, produce this change,

each having a particular angle at which the effect is the greatest. The *angle of polarization* of a substance is, then, the angle that the incident ray must make with the perpendicular to a plane surface of that substance in order that the polarization be complete. For glass this angle is 56° 45′; for water, 52° 45′; for quartz, 57° 32′; for diamond, 68°; etc.

If in the preceding experiment the plane had been inclined at a larger or smaller angle, the ray of light would not have been completely polarized, which would be shown by the ray being partially reflected from the analyzing plate *B B* in *all* positions.

161. Polarization in Opposite Directions.—The ordinary and extraordinary rays produced by sending a ray of light through a substance possessing the property of double refraction are found on examination to be completely polarized, but in a very peculiar manner. The one is capable of reflection when the other is transmitted or refracted. This fact is expressed by saying that the two rays are *polarized in opposite directions*. With a good sized piece of transparent Iceland spar, the two oppositely polarized rays may be widely separated and examined separately by an analyzing plate.

162. Some of the doubly refracting substances absorb one of these rays, but not the other. *Tourmaline*, a mineral, is the best known medium possessing this property. When a ray of light is sent through a plate of tourmaline cut at right angles to the axis, an ordinary and an extraordinary ray are formed, which are polarized in planes at right angles to each other. The mineral, however, rapidly absorbs the ordinary ray, and if the emerging polarized extraordinary ray is observed through a second similar plate, held exactly in the same position as the first one and as shown in Fig. 34 (*a*), little change will be noticed; but when the second plate is turned around as shown in Fig. 34 (*b*), so as to make the axes cross at right angles, it will be

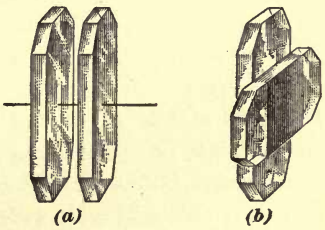

FIG. 34.

found that the light is almost wholly stopped. A plate of this mineral in this way is a valuable test for detecting polarized light, and light that has not undergone the change. Instead of tourmaline plates, which are, owing to the thickness required for complete polarization, not very transparent, and are mostly more or less colored, two Nicol's prisms, or the cheaper and quite as effective Foucault's prisms, are used, as they possess the same properties as tourmaline.

Two of these prisms, placed in exactly the same position, one behind the other, allow light to pass through them; but by turning the second prism a cloudiness is produced, until, by continuing to turn the prism, a point is reached that causes perfect darkness. This phenomenon occurs whether the light is colored or not.

163. Nicol's prism consists of a bar of Iceland spar, about 1 inch in height and $\frac{1}{3}$ inch in breadth; the prism is cut in the plane that passes through the obtuse angles, as shown in Fig. 35 (*a*). The two halves are joined by a layer

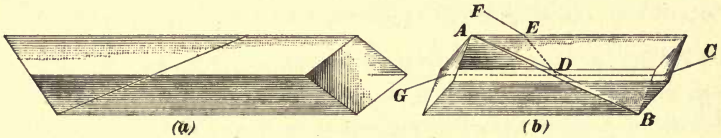

FIG. 35.

of Canada balsam. The principal involved in the construction of this prism is the following: The refractive index of Canada balsam, being 1.549, is less than the index of the ordinary ray of Iceland spar, 1.654, and greater than the index of its extraordinary ray, 1.483; hence, if a ray of light *C D*, Fig. 35 (*b*), enters the prism, the ordinary ray is totally reflected on the surface *A B* and takes the direction *D E F*, by which it is refracted out of the prism, while the extraordinary ray *D G* emerges alone. The prism is used as an analyzer and also as a polarizer, possessing the same properties as tourmaline, but, as has been already mentioned, is preferable because it gives a colorless field of view.

In **Foucault's prism,** which is constructed on exactly the same principle, the layer of Canada balsam is replaced by a layer of air, the prism being held together by a brass or aluminum mounting.

164. Circular Polarization.—A ray of light colored by passing, for example, through a plate of red glass, which has passed through the first of two Nicol's prisms, and which is then obstructed in consequence of the position of the second prism, regains its original property of penetrating the second prism if a plate of rock crystal is placed between the two prisms in the path of the ray. The passage of the ray through the analyzer (second prism) can now only be interrupted by a further turn of the analyzer. The rotation required depends on the thickness of the plate and the color of the ray. It increases from red in the following order: yellow, green, blue, and violet.

This kind of polarization is known as *circular polarization*, and for a long time quartz or rock crystal was the only solid substance known to exhibit this peculiar property of producing it. Scientific investigations and experiments, however, have brought to light since then a number of substances that possess this property in a still higher degree; thus, for instance, a plate of cinnabar acts nearly fifteen times more powerfully than a plate of equal thickness of rock crystal.

The direction of the rotation that is required to produce this complete opaqueness of the analyzer depends on the composition of the interposed substance. A substance is said to be *dextro-* or *levo-rotary*, according as it is necessary to turn the analyzer around in the direction of the hands of a watch, or in the opposite direction. The angle through which the polarizer has to be turned is called the *angle of rotation.*

165. It has been observed that solutions of many organic substances show the property of circular polarization, though, as a rule, to a less extent than rock crystal. Thus, solutions of cane sugar and tartaric acid possess right-handed polarization, while oil of turpentine and albumin are left-handed. In all of these solutions the angle of rotation varies directly

with the strength of the solutions, and the thickness of the column through which the light passes; hence, circular polarization has become an important auxiliary in chemical analysis.

This power of circular polarization is found most commonly in vegetable and animal products and their immediate derivatives, as will be seen in the description of such bodies in *Organic Chemistry*. It is assumed that this power depends on some peculiarity in the structure of the molecules.

166. In order to determine the amount of polarization any liquid may exhibit, it is poured into a glass tube and closed with glass plates. This is then placed between two Nicol's prisms, which have previously been so arranged that the analyzer appears opaque. Through the polarizing power of the liquid, the analyzer becomes transparent, and from the magnitude of the angle through which the analyzer has to be turned to appear opaque again, the polarizing power of the liquid is calculated.

167. An apparatus of this kind is widely used and is of the greatest importance in the sugar industry. It is called the **saccharimeter,** and is used for determining the concentration of solutions of cane sugar. The instrument was devised by *Biot*, and through certain additions perfected by *Soleil.*

Though a full description of the saccharimeter, its uses, and instructions in regard to calculations, etc. cannot be given in this section, a short explanation of this apparatus will enable the student to comprehend more fully the importance of circular polarization in connection with chemistry, and will give him some idea as to the way in which this physical phenomenon is utilized.

This apparatus is shown in Fig. 36. The two Nicol's prisms are in the fastenings A and B, respectively. After the prisms are arranged so that the analyzer A appears opaque, the tube, filled with a solution of sugar, is placed as marked by dotted lines in the figure. The light will now

be again enabled to pass through the analyzer, and prism A has to be turned through a certain angle till entire darkness again appears. The magnitude of this angle is read on the circular disk C, which is divided into degrees, and upon which moves an index D connected with the prism A.

If we assume that the tube used is exactly 10 inches long and the solution to be analyzed contains exactly 10 per cent.

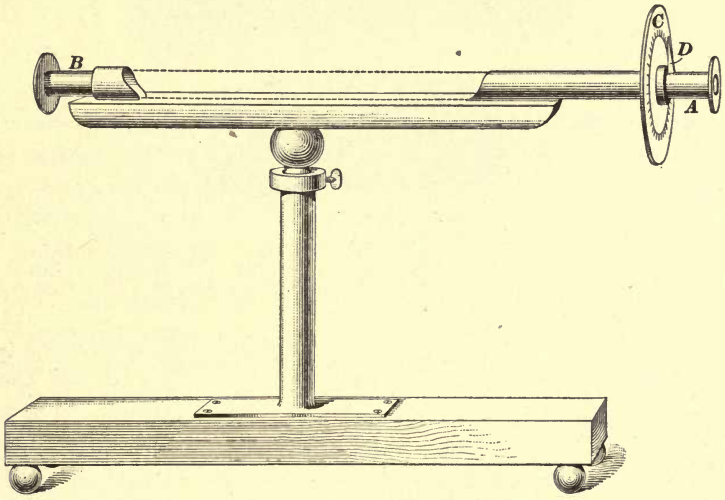

FIG. 36.

of cane sugar, the angle of rotation will be 19.6°. Now, the magnitude of this angle is directly proportional to the length of the liquid column and also to the percentage of sugar in solution. If, therefore, a solution containing x per cent. of sugar in a tube y inches long produces an angle of rotation of z degrees, the percentage of sugar will be given by the equation:

$$\frac{z}{19.6} = \frac{y}{10} \times \frac{x}{10}, \text{ or, } x = \frac{100z}{19.6y}. \qquad (29.)$$

The apparatus shown in Fig. 36 is of the simplest kind, and can only be used conveniently when pure cane sugar is to be analyzed; if the solution contains cane sugar and uncrystallizable sugar, the latter possessing left-handed

rotation and the former right-handed, only the difference of these two actions is obtained. The whole quantity has to be changed into uncrystallizable sugar, the experiment repeated, and from the results of the two observations the quantity of each kind of sugar calculated. Soleil has added to this form of saccharimeter a number of improvements that facilitate observations and make measurements more exact.

MAGNETISM.

168. **Magnets** are substances that have the property of attracting pieces of iron or steel, and the term *magnetism* is applied to the cause of this attraction. Magnetism exists in a natural state in an ore of iron, which is known in chemistry as *magnetic oxide of iron* or *magnetite*. This magnetic ore was first found by the ancients in *Magnesia*, a city in Asia Minor; hence, the name *magnet*.

It was also discovered that when a small bar of this ore is suspended in a horizontal position by a thread, it has the property of pointing in a north-and-south direction. From this fact the name *lodestone*, "leading stone," was given to the ore.

When a bar or needle of hardened steel is rubbed with a piece of lodestone, it acquires magnetic properties similar to those of the lodestone, without the latter losing any of its own force. Such bars are called *artificial* magnets.

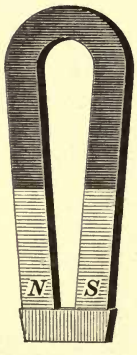

FIG. 37.

Artificial magnets that retain their magnetism for a long time are called *permanent* magnets.

The common form of artificial or permanent magnet, Fig. 37, is a bar of steel bent into the shape of a *horseshoe* and then hardened and magnetized. A piece of soft iron called an *armature*, or a *keeper*, is placed across the two free ends and helps to prevent the magnet from losing its magnetic force.

169. If a bar magnet is dipped into iron filings, the filings are attracted towards the two ends and adhere there in tufts; while towards the center of the bar, half-way between the two ends, there is no such tendency (see Fig. 38). That part of the magnet where there is no apparent magnetic attrac-

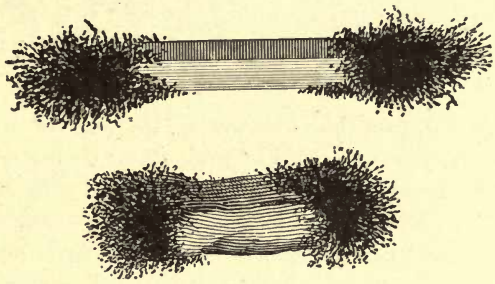

FIG. 38.

tion is called the *neutral line;* and the parts around the ends where the attraction is greatest are called *poles.* An imaginary line drawn through the center of the magnet from end to end, connecting the two poles together, is the *axis of magnetism.*

170. A **compass** consists of a magnetized steel needle, Fig. 39, resting upon a fine point and capable of turning freely in a horizontal plane. When not in the vicinity of other magnets or magnetized iron, the needle will always come to rest with one end pointing towards the north and the other towards the south. The end pointing northward is the *north-seeking pole*, or, simply, the *north pole*, and the opposite end is the *south-seeking pole*, or *south pole.* This *polarity* applies as well to all magnets.

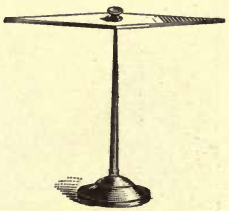

FIG. 39.

If the north pole of one magnet is brought near the south pole of another magnet, attraction takes place; but if two north poles or two south poles are brought together, they repel each other. In general, *like magnetic poles repel each other; unlike poles attract each other.*

171. The earth is a great magnet whose magnetic poles coincide nearly, but not quite, with the true geographical north and south poles. A freely suspended magnet, therefore, will always point in a north-and-south direction.

172. It is impossible to produce a magnet with only one pole. If a long bar magnet is broken into any number of parts, each part will still be a magnet and have two poles —one north and one south pole.

173. Substances that are not in themselves magnets—that is, do not possess poles and neutral lines, but are capable of being attracted by a magnet—are called *magnetic substances.* In addition to iron and its alloys, the following metals are magnetic substances: *nickel*, *cobalt*, *manganese*, *cerium*, and *chromium.* These metals, however, possess magnetic properties in a very inferior degree compared with iron and its alloys. All other known substances are called *non-magnetic substances.*

174. The space surrounding a magnet is called a *magnetic field;* or, in other words, a magnetic field is a place where a freely suspended magnetic needle will always come to rest, pointing in the same direction.

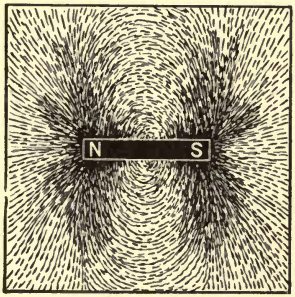

FIG. 40.

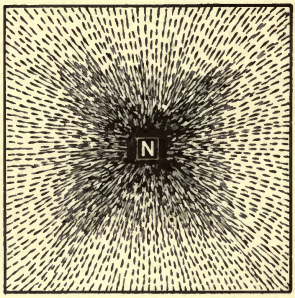

FIG. 41.

175. Magnetic attractions and repulsions are assumed to act in a definite direction and along imaginary lines called *lines of magnetic force*, or, simply, *lines of force*, and every magnetic field is assumed to be traversed by such lines of

force. Their position in any plane may be shown by placing a sheet of paper over a magnet and sprinkling fine iron filings over the paper. In the case of a bar magnet lying on its side, the iron filings will arrange themselves in curved lines extending from the north to the south pole, as shown in Fig. 40. A view of the magnetic field looking towards either pole of a bar magnet would exhibit merely radial lines, as shown by the filings in Fig. 41.

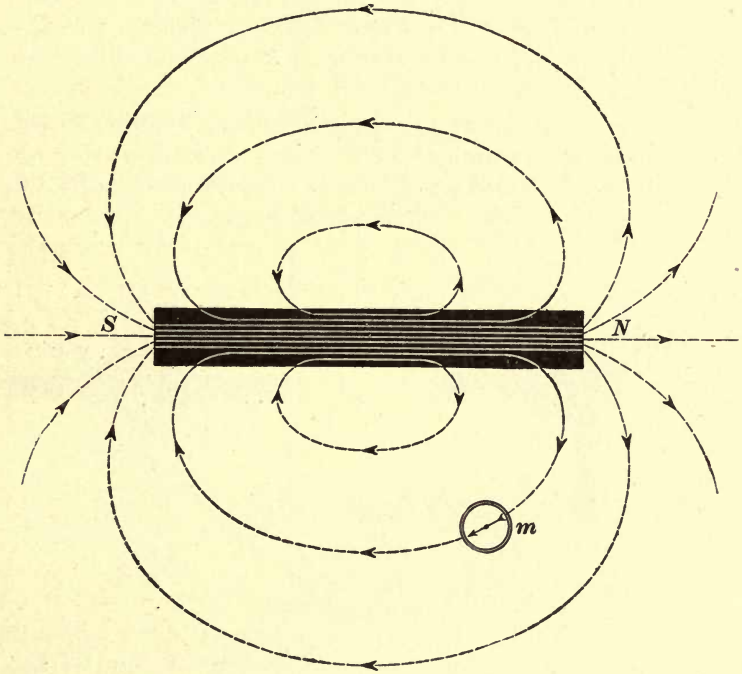

FIG. 42.

Every line of force is assumed to pass out from the north pole, make a complete circuit through the surrounding medium, and into the south pole, thence passing through the magnet to the north pole again, as shown in Fig. 42. This

NOTE.—In all diagrams, the *direction of the lines of force* will be represented by arrowheads upon dotted lines.

is called the *direction of the lines of force*, and the path they take is called the *magnetic circuit.*

The direction of the lines of force in any magnetic field can be traced by a small, freely suspended magnetic needle or a small compass, such as indicated by *m* in Fig. 42. The north pole of the needle will always point in the direction of the line of force, the length of the needle lying tangent to the lines of force at that point. If the needle is moved bodily in the direction towards which the north pole points, its center or pivot will describe a path coinciding with the direction of the lines of force along that part of the magnetic field.

Lines of force can never intersect; when two opposing magnetic fields are brought together, as indicated by the iron filings in Figs. 43 and 44, the lines of force from each will

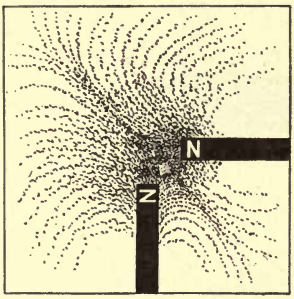

FIG. 43.

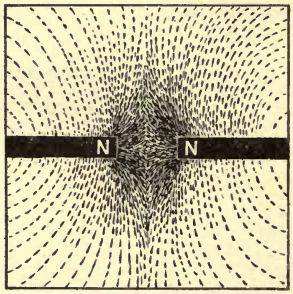

FIG. 44.

be crowded and distorted from their original direction until they coincide in direction with those opposing and form a resultant field in which the direction of the lines of force will depend on the relative strengths of the two opposing negative fields. The resulting poles thus formed are called *consequent poles.*

In every magnetic field there are certain stresses that produce a *tension* along the lines of force and a *pressure* across them ; that is, they tend to *shorten* themselves from end to end, and *repel* one another as they lie side by side.

176. When a magnetic substance is brought into a magnetic field, the lines of force in that vicinity crowd together and all tend to pass through the substance. If the substance is free to move on an axis, but not bodily towards the magnet pole, it will always come to rest with its greatest extent or length in the direction of the lines of force. The body will then become a temporary magnet, its south pole being situated where the lines of force enter it, and its north pole where they pass out. The production of magnetism in a magnetic substance in this manner is called *magnetic induction.* The production of artificial magnetism in a hardened-steel needle or bar by contact with lodestone is only a special case of magnetic induction.

177. The *amount*, or *quantity*, *of magnetism* is expressed by the total number of lines of force contained in a magnetic circuit.

178. *Magnetic density* is the number of lines of force passing through a unit area measured perpendicularly to the direction of the lines.

ELECTRICITY.

INTRODUCTION.

179. **Electricity** is the name given to that which directly causes all electrical phenomena. The word is derived from the Greek word *ἤλεκτρον*, meaning "amber."

Although electrical science has made great advances in the last few years, the exact nature of electricity is unknown. Recent researches tend to demonstrate *that all electrical phenomena are due to a peculiar state or stress of a medium called ether;* that, when in this condition, the *ether* possesses *potential energy*, or *capacity for doing work*, as is

manifested by attractions and repulsions, by chemical decomposition, and by luminous heating, and various other effects.

180. In all probability, electricity is not a form of *matter*, for the only physical properties it possesses in common with material substances are *indestructibility* and *elasticity;* it does not possess *weight*, *extension*, nor any of the other physical properties of matter.

181. Electrical science is founded upon the effects produced by *the action of certain forces upon matter*, and all knowledge of the science is deduced from these effects. The study of the fundamental principles of the science is an analysis of a series of experiments and the classification of the results, under laws and rules. It is not necessary to keep in mind any hypothesis as to the exact nature of electricity; its effects and the laws that govern them are quite similar to those of well known mechanical and natural phenomena, and will be best understood by comparison.

182. Electricity may either appear to reside upon the surfaces of bodies as a *charge*, under high pressure, or flow through their substance as a *current*, under comparatively low pressure. That branch of science which treats of charges upon the surfaces of bodies is termed *electrostatics*, and the charges are said to be *static charges.*

Electrodynamics is that branch which treats of the action of *electric currents.*

ELECTROSTATICS.

183. When a glass rod or a piece of amber is rubbed with a piece of silk or fur, the parts rubbed will be found to have the property of attracting light bodies, such as pieces of silk, wool, feathers, gold leaf, pith, etc., which, after momentary contact, are repelled. These attractions and repulsions are caused by a *static charge of electicity*, residing upon the surfaces of those bodies. A body in this condition is said to be *electrified.*

A better experiment for demonstrating this action is to suspend a small pith ball by a silk thread from a support or bracket, as shown in Fig. 45. Such an apparatus is spoken of as an *electric pendulum.* If a static charge of electricity is developed on a glass rod by rubbing it with silk, and the rod brought near the pendulum, the ball will be attracted to the rod; but, after momentary contact, it will be repelled. By this contact the ball becomes electrified, and, so long as the two bodies retain their charges, mutual *repulsion* will take place whenever they are brought near each other. If a stick of sealing wax, electrified by being rubbed with fur, is brought near to another pendulum, the same results will be produced—the ball will fly towards the wax, and, after contact, will again be repelled. But the charges respectively developed in the preceding cases are not in the same condition. For if, after the pith ball has been touched with the glass rod and repelled, the electrified sealing wax is brought in the vicinity, *attraction* takes place between the ball and the sealing wax. Similarly, if the pendulum is charged with the electrified sealing wax, the ball will be repelled by the wax and attracted by the glass rod.

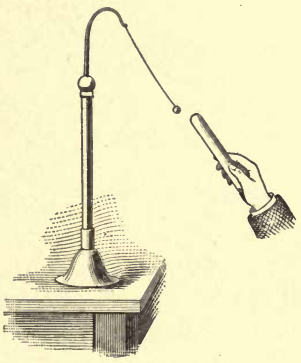

FIG. 45.

An electric charge excited upon glass by rubbing it with silk has been termed a *positive charge* (+); and that developed on resinous bodies by friction with flannel or fur, a *negative charge* (−).

184. Neither charge is produced alone, for there is always an equal quantity of both charges produced, one charge appearing on the body rubbed, and an equal amount of the opposite charge upon the person or thing rubbing.

185. The *intensity of the charge* developed by rubbing the two substances together is independent of the *actual*

amount of friction that takes place between the bodies. For, in order to obtain the highest possible degree of electrification, it is only necessary to bring every portion of one surface into intimate contact with every particle or every portion of the other surface; when this is done, no extra amount of rubbing can develop any greater charge upon either substance.

186. From these experiments are derived the following laws:

1. *When two dissimilar substances are placed in contact, one of them always assumes the positive and the other the negative condition, although the amount may sometimes be so small as to render its detection very difficult.*

2. *Electrified bodies with similar charges are mutually repellent, while electrified bodies with dissimilar charges are mutually attractive.*

187. In the following list, called the *electric series*, the substances are arranged in such order that each receives a *positive* charge when rubbed with any of the bodies following, and a *negative* charge when rubbed with any of those preceding it.

1. Fur	6. Cotton	11. Sealing wax
2. Flannel	7. Silk	12. Resin
3. Ivory	8. The body	13. Sulphur
4. Crystals	9. Wood	14. Gutta percha
5. Glass	10. Metals	15. Guncotton

For example, *glass*, when rubbed with *fur*, receives a *negative* charge; but when rubbed with *silk* it receives a *positive* charge.

188. An **electroscope** is an instrument for detecting static charges of electricity, and for determining whether they are positive or negative; it does not measure the intensity of the charges.

The pith ball suspended by a silk thread acts as a simple elec-

troscope. A more sensitive electroscope is shown in Fig. 46, and consists of two gold leaves suspended within a glass jar *J*, which serves to protect them from drafts of air and to support them from contact with the earth. The gold leaves *a* are supported side by side in the jar by a brass rod or wire *b*, which passes through a cork in the mouth of the jar. The upper end of the brass rod is furnished with a flat metallic plate or ball *c*. An electrified body, such as the rod *d*, brought into the vicinity of the electroscope will cause the leaves to repel each other, and they will remain in that position for some time.

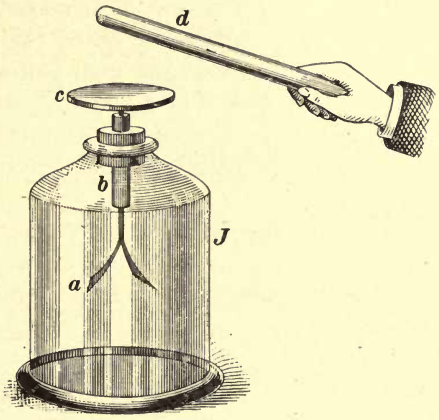

FIG. 46.

To determine the condition of a charge by the electroscope: First, charge the gold leaves with a known charge, such as that developed upon glass when rubbed with silk. The leaves will spread apart, being electrified with a *positive* charge. When they are thus charged, the approach of a body *positively* charged will cause them to open still more widely; while on the approach of one *negatively* charged, they will close together.

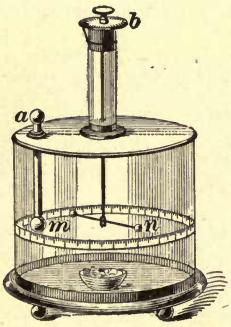

FIG. 47.

189. The **torsion balance** is an instrument used to measure the *force* exerted between two electrified bodies.

It consists of an arm or lever of some light insulating material, such as a straw or piece of wood, provided at one end with a gilt pith ball *n*, Fig. 47, and suspended in a glass jar by a fine silver wire. The wire passes

up through a glass tube and is fastened to a brass stopper *b*, called the *torsion head.* The torsion head is graduated in degrees and is capable of being revolved around upon the glass tube. Another gilt pith ball *m* is fastened to the end of the vertical glass rod *a*, which is inserted through an opening in the top of the jar. A narrow strip of paper, also divided into degrees, encircles the glass jar at the level of the two pith balls.

To use the torsion balance: Turn the torsion head around until the two pith balls *m* and *n* just touch each other. Remove the glass rod *a*, and communicate the charge to be measured to the gilt ball *m*. Replace the glass rod in the jar. The two gilt balls will touch each other momentarily and half of the charge will pass from *m* to *n*. As both balls possess similar charges, they will immediately repel each other; the ball *n* being driven around, twists up the wire to a certain extent. The resistance of the wire to torsion will eventually balance the force of repulsion, and the ball *n* will come to rest, the balls being separated by a certain distance. In any wire, the force of torsion is proportional to the amount of twist, or, in this case, to the angle of torsion; hence, the force exerted between the two balls can be measured by the angle described by the ball *n*.

190. By means of the torsion balance, it is proved *that the force exerted between two bodies, statically charged with electricity, varies inversely as the square of the distance between them.*

Thus, suppose two electrified bodies $\frac{1}{4}$ inch apart repel each other with a certain force; at a distance of 1 inch, the force would only be *one-sixteenth* as great. This law is equally true for the force of attraction between two bodies with dissimilar charges.

A unit quantity of electricity is that charge which, when placed in air at a distance of 1 *centimeter* from another equal and similar charge, will be repelled with the force of 1 *dyne.*

NOTE.—(*a*) The *centimeter*, or the unit of length, represents $\frac{1}{1{,}000{,}000{,}000}$ of the distance from the pole to the equator on the surface

of the earth, and is equal to .3937 inch, or 1 inch equals 2.54 centimeters, nearly. A *square centimeter* is the area contained in a square each of whose sides is 1 centimeter in length; 1 square centimeter equals .155 square inch, or 1 inch equals 6.45 square centimeters, nearly. A *cubic centimeter* is the volume contained in a cube each of whose edges is 1 centimeter in length; 1 cubic centimeter equals .06102 cubic inch, or 1 cubic inch equals 16.387 cubic centimeters.

(*b*) The *gram*, or the unit of mass or quantity of matter, represents the quantity of matter contained in a cubic centimeter of pure water at the temperature of its maximum density, which is 4° C., or 39.2° F., and is equal in weight to 15.432 grains.

(*c*) The *second*, or the unit of time, represents $\frac{1}{86,400}$ of an ordinary day of 24 hours.

This system of units is designated as the *absolute* or C. G. S. system, to distinguish it from other systems based upon other fundamental units.

The secondary units, derived from these fundamental units, are the *unit of velocity* and the *dyne*.

The unit of *velocity*, or the rate at which a body changes its relative position, is determined by dividing the distance in centimeters through which a body travels by the time in seconds required to travel that distance. The unit of velocity is, therefore, 1 centimeter per second. *To compute the velocity in centimeters per second, divide the number of centimeters by the number of seconds.*

The *dyne*, or the unit of force, is that force which, by acting upon a mass of 1 gram for 1 second, can give to it a velocity of 1 centimeter per second.

191. In either case, whether of attraction or repulsion, the force at any given distance is equal to the product of the two quantities of electricity on the bodies. That is, if a certain body were charged with 4 *unit quantities* of electricity, and another with 3 *unit quantities*, then the force exerted between them would be 12 times as great as if each had contained a charge of 1 *unit*.

192. Only that part of a dry glass rod which has been rubbed will be electrified; the other parts will produce neither attraction nor repulsion when brought near an electroscope. The same is true of a piece of sealing wax or resin. These bodies do not readily *conduct* electricity, that is, they oppose or resist the passage of electricity through them. Therefore, electricity can reside only as a *charge* upon that part of their surfaces where it is developed. Experiments show that when a metal receives a charge at any point, the electricity immediately passes or flows through its substance to all parts. Metals, therefore, are said to be

good conductors of electricity. Bodies have, accordingly, been divided into two classes; namely, *non-conductors* or *insulators*, those bodies that offer an infinitely high *resistance* to the passage of electricity; and *conductors*, or those that offer a comparatively low resistance to its passage. This distinction is not absolute, for all bodies conduct electricity to some extent, while there is no known substance that does not offer some resistance to its flow.

193. *Electrical resistance* may be defined as a general property of matter, varying with different substances, by virtue of which matter *opposes* or *resists* the passage of electricity.

194. *Conductivity* is the facility with which a body transmits electricity, and is the opposite of resistance. For instance, copper is of low resistance and high conductivity; wood is of high resistance and low conductivity.

195. In giving the following list and dividing the different substances into two classes, it should be understood that it is done only as a guide for the student. Between these classes are many substances that might be included in either, and no hard-and-fast line can be drawn. The list is arranged in order of the *conductivity* of the different substances, beginning with silver, which is the best known conductor.

Conductors	Non-Conductors or Insulators
Silver	Paper
Copper	Oils
Other metals	Porcelain
Charcoal	Wood
Ordinary water	Silk
	Resins
	Gutta percha
	Shellac
	Ebonite
	Paraffin
	Glass
	Dry air

196. An electric charge will be *induced* in a conductor when that conductor is brought into the vicinity of an electrified body. This effect is termed *electrostatic induction*, and the range of space in which it can take place is an *electrostatic field.*

If the conductor *A B*, Fig. 48, is supported from contact with the earth by insulators, and is then brought into the electrostatic field of the conductor *C*, but not touching *C*, which is electrified with a *positive* charge, the following facts will be observed:

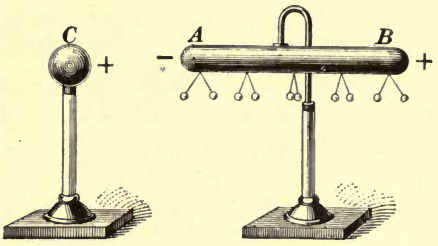

FIG. 48.

1. A charge will be produced on *A B*, as is shown by the pith balls spreading apart.

2. This charge will be *negative* at the end *A* nearest *C*, and *positive* at the end *B* farthest from *C*, as can be shown by an electroscope.

3. The charges at *A* and *B* are equal to each other; for, if the conductor *A B* is removed from the vicinity of the conductor *C* without having touched *C*, the opposite charges immediately neutralize each other; that is, no electrification will be indicated by the pith balls.

4. Again, as *C* is brought nearer and nearer *A*, the charges on opposite signs on the approaching surfaces attract each other more and more strongly until *C* is brought very near *A*, when a spark darts across the intervening space. The two charges, rushing together, neutralize each other, leaving the induced *positive* charge, which was formerly repelled to the end *B* of the conductor, as a permanent charge over all the surface of *A B*.

5. Or, if the conductor *A B* is touched by a conductor connected to the earth when it is under the influence of *C*, the *positive* charge will be neutralized by the earth, and the *negative* charge will remain when *A B* is removed from the field of *C*. The charge that passes to the earth from *A B* is

called a *free charge*, while that charge which is held by the inductive influence of C is a *bound charge*. Both free and bound charges can be *negative* or *positive*, depending upon the sign of the charge on C.

When two conducting bodies, both electrified with equal dissimilar charges, are touched together momentarily, the two charges will neutralize each other, no trace of either remaining; but if they are unequal, the smaller charge will neutralize an equal amount from the larger and leave a charge that is equal to the difference between the two original charges, the sign of the remaining charge being the same as that of the larger one. Before the bodies can be separated, the remaining charge will divide equally between the two bodies. For example, two gilt balls A and B are charged respectively with $+20$ and -4 units of electricity. When the balls are placed in contact, the -4 charge on B will neutralize a $+4$ charge on A and leave a $+16$ charge, which immediately divides equally between the two balls; that is, a charge of $+8$ units remains on each ball when they are separated.

It is found that the effect of this electrostatic induction is greatly increased by placing some other substance, such as glass or paper, between the two bodies.

197. The facility with which a body allows electrostatic induction to act across it is called its *inductive capacity*. The inductive capacity varies with different substances, but almost all non-conductors are better than air.

198. The **electrophorus**, Fig. 49, is an instrument devised for the purpose of obtaining an almost unlimited number of static charges of electricity from one single charge, and is based upon the principle of electrostatic induction.

It consists of two main parts: a thin cake of resinous material cast in a round metal dish or pan B, about 1 foot in diameter; and a round disk A of slightly smaller diameter, made of metal and provided with a glass handle. In using

the electrophorus, the resinous cake must first be beaten or rubbed with a warm piece of woolen cloth or fur. The disk, or cover, is then placed on the cake, touched momentarily with the finger to liberate the *free* charge, then removed by taking up by the handle. It is now found to be powerfully electrified with a *positive* charge; so much so, indeed, as to yield a considerable spark when the hand is

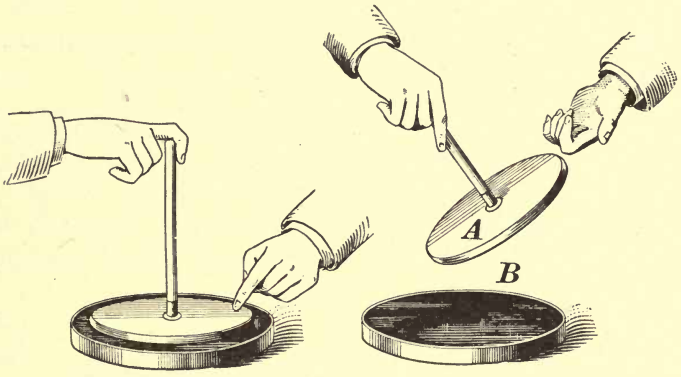

FIG. 49.

brought near it. The cover may be replaced, touched, and again removed, and will thus yield any number of sparks; the original charge on the resinous plate meanwhile remaining practically as strong as ever.

199. A static charge of electricity is not usually distributed uniformly over the surface of conducting bodies. Experiments show that there is more electricity on the edges and corners than upon their flatter parts.

The term *electric density* is used to signify the quantity of electricity residing in a small area at any part of a body, the distribution being supposed to be uniform over that small part of the surface.

The electric density is the quotient arising from dividing the total charge of electricity in units of quantity residing upon the surface of a body, by the area of the surface in square inches. For example, a charge of 240 units of electricity is

imparted to a sphere, the surface area of which is 40 square inches; then, the electric density over the surface of the sphere is $\frac{240}{40} = 6$ units of electricity per square inch.

200. Electrostatic machines have been devised for the purpose of obtaining larger static charges than can be developed by rubbing the glass rod or by the electrophorus. They consist, mainly, of two parts, one for producing and the other for collecting the charges.

There are three important kinds of electrostatic machines: the *cylinder*, the *plate*, and the *induction* machines.

201. The *cylinder machine*, as usually constructed, consists of three principal parts: a cylinder of glass, revolving upon a horizontal axis; a rubber, or cushion of horsehair, to which is attached a long silk flap; and an insulating metallic cylinder called a *prime conductor*. In Fig. 50, the

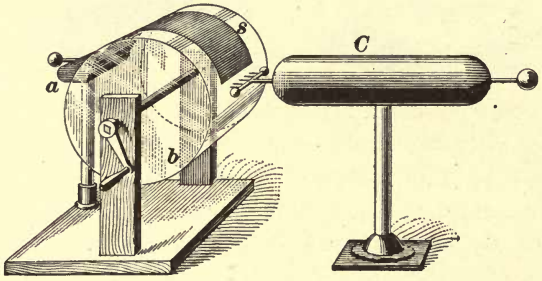

FIG. 50.

cushion of horsehair *a*, covered with a coating of amalgam of zinc, presses against the glass cylinder *b* from behind, allowing the silk flap *s* to rest upon the upper half of the glass. The prime conductor *C* is provided at one end with a row of fine metallic spikes and is placed in front of the machine with the row of spikes projecting towards the glass cylinder. When the glass cylinder is rotated, a positive charge is produced upon the glass and a negative charge upon the rubber. The positive charge is carried around upon the glass cylinder, and, just before reaching a position opposite the row of spikes, it acts inductively upon the prime conductor,

attracting a negative charge to the near end and repelling a positive charge to the far end. When the positive charge arrives in front of the row of spikes, it will be neutralized by the attracting negative charge from the conductor, leaving the glass in a neutral condition, ready to be excited again. A positive charge now remains upon the prime conductor and can be utilized for other experiments.

202. The *plate machine* is similar in all respects to the cylinder machine with the exception that a glass or ebonite plate is used instead of the glass cylinder, and there are usually two sets of rubbers or cushions, instead of one. Each set of cushions is double; that is, it is made in two parts, with the plate revolving between them. One set of cushions is placed at the top of the machine, and the other at the bottom, with silk flaps extending from each over a quadrant of the plate. The charge is collected on two prime conductors connected by a metal rod, and each is provided with a row of fine spikes at one end. They are placed in such a position that the two rows of fine spikes project towards the glass plate at opposite sides of its horizontal diameter. The electrostatic action of the machine is in all respects the same as that of the cylinder machine.

203. The *induction machine* differs widely in its action from the two machines previously described. It requires an initial charge from some exterior source to start its action. The initial charge acts inductively across a revolving glass plate, and produces other charges; these charges in turn are conveyed by the moving parts to some other point, where they increase the initial charge, or furnish a supply of electricity to a prime conductor.

The two principal machines of this class are the *Holtz* and the *Wimshurst.*

204. It has been shown that opposite charges attract and hold one another; that electricity cannot flow through glass, and yet can act across it by induction. If a piece of tin foil is pasted upon the middle of each face of a thin plate

of glass, and one of the pieces is electrified with a positive charge and the other with a negative charge, the two charges will attract each other; or, in other words, they are held, or *bound*, by each other. It will be found that these two pieces of tin foil may be charged a great deal stronger in this manner than either of them could possibly be if they were stuck to the glass alone and then electrified. This property of retaining and accumulating a large quantity of static charges, which two conductors possess when placed side by side and separated from each other by a non-conductor, is termed *capacity*.

205. A **condenser** is an apparatus for *condensing* or accumulating a large quantity of static charges of electricity on a comparatively small surface. It consists of two conductors separated by a thin layer of some non-conducting material; one of the plates is entirely insulated from the earth, and the other is connected to it by a conductor.

The capacity of a condenser depends upon the size and form of the condensing plates, the thinness of the insulating material between them, and the *inductive capacity* of the insulating material.

206. A convenient form of condenser is called the *Leyden jar*, Fig. 51. It consists of a glass jar *J* coated up to

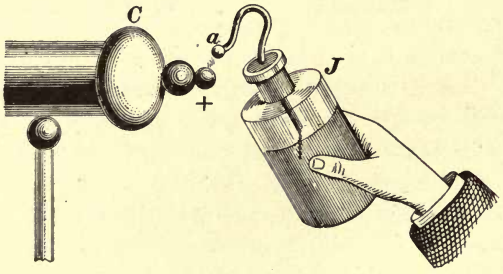

FIG. 51.

a certain height on the inside and outside with tin foil. A brass knob *a* is fixed on the end of a stout brass wire, which passes downwards through a lid or stopper of dry, well

varnished wood, and connected by a loose piece of brass chain with the inner coating of the jar.

To charge the jar, the knob is held to the prime conductor *C* of an electrical machine, the jar being either held in the hand by the outer tin foil coating or connected to the earth by a wire or chain. When a positive charge is thus imparted to the inner coating, it acts inductively on the outer coating, attracting a negative charge in the face of the outer coating nearest the glass, and repelling a positive charge to the outside of the outer coating. This outer charge then passes through the hand or any conductor to the earth.

207. An **electrostatic battery** consists of a number of Leyden jars whose inside coatings are all connected together and whose outside coatings are all connected to the earth.

ELECTRODYNAMICS.

208. In dealing with electric currents the word *potential* will be substituted for the general and vague phrase *electrical condition.*

The term potential, as used in electrical science, is analogous to *pressure* in gases, *head* in liquids, and *temperature* in heat.

When an electrified body, *positively* charged, is connected to the earth by a conductor, electricity is said to *flow from* the body *to* the earth; and, conversely, when an electrified body *negatively* charged is connected to the earth, electricity is said to flow *from* the earth *to* that body. That which determines the *direction of flow* is the relative electrical *potential* or *pressure* of the two charges with respect to the earth.

209. It is impossible to say with certainty in which direction electricity really flows, or, in other words, to declare which of two points has the higher and which the

lower electrical potential, or pressure. All that can be said with certainty is, that when there is a *difference* of *electrical potential*, or *pressure*, an electric current tends to flow *from* the point of higher *to* that of lower potential, or pressure.

For convenience, it has been arbitrarily assumed and universally adopted, that that electrical condition called positive is at a *higher potential*, or *pressure*, than that called negative; and that an electric current flows *from* a positively *to* a negatively electrified body.

210. The zero, or normal, level of water is taken as that of the surface of the sea, and the normal pressure of air as that of the atmosphere at the sea level; similarly, there is a *zero pressure*, or *potential*, of electricity in the earth itself. It may be regarded as a reservoir of electricity of infinite quantity, and its pressure, or potential, taken as zero. In Art. **208,** the condition called positive is assumed to be at a higher potential than the earth, and that called negative is assumed to be at a lower potential than the earth.

It must be understood that electricity is a condition of matter and not matter itself, for it possesses neither weight nor extension. Consequently, the statement that electricity is flowing through a conductor must not be taken too literally; it must not be supposed that any material substance, such as a liquid, is actually passing through the conductor in the same sense as water flows through a pipe. The statement that electricity is flowing through a conductor is only another way of expressing the fact that the conductor and the space surrounding it are in different conditions than usual, and that they possess unusual properties. The action of electricity, however, is quite similar in many respects to the flow of liquids, and the study of electric currents is much simplified by the analogy.

211. *In order to produce what is called an electric current, it is first necessary to cause a difference of electrical potential, or pressure, between two bodies or between two parts of the same body.*

In Art. **186,** it was stated that when two dissimilar substances are simply placed in contact, one always assumes the positive and the other the negative condition; in other words, a *difference of electrical potential* is developed between the two bodies.

Placing a piece of copper and zinc in contact will develop a difference of electrical potential, which can easily be detected. The same results will follow if the plates are slightly separated from each other and placed in a vessel containing saline or acidulated water, leaving a small portion of one end of each plate exposed. The exposed ends of the zinc and copper are now electrified to different degrees; or, in other words, there is a difference of electrical potential between the plates, one plate being at a higher potential than the other.

When the exposed ends are connected together by any conducting material, the potential between the plates tends to equalize, and a momentary rush or discharge of electricity passes between the exposed ends through the conducting material and between the submerged ends through the liquid. During its passage through the liquid, the electricity causes certain chemical changes to take place; these chemical reactions cause in their turn a fresh difference of potential between the plates, which is followed immediately by another equalizing discharge, and that by a further difference, and so on. These changes follow one another with great rapidity; so rapid, in fact, that it is impossible to distinguish them, and they appear absolutely *continuous.* The equalizing flow that is constantly taking place from one plate to the other is known as a *continuous current* of electricity. Consequently, *an electric current becomes continuous when the difference of potential is constantly maintained.*

By the use of a very delicate electroscope, the exposed end of the copper will be found to be electrified with a positive charge, and the submerged end with a negative charge; in the case of the zinc, the opposite conditions exist, namely, the exposed end is electrified with a negative charge, and the submerged end with a positive charge. The current,

therefore, will flow *from* the exposed end of the copper through the conductor *to* the exposed end of the zinc, and *from* the submerged end of the zinc through the liquid *to* the submerged end of the copper.

212. A **simple voltaic** or **galvanic cell** is shown in Fig. 52. It is an apparatus for developing a continuous current of electricity, and consists, essentially, of a vessel *A* containing saline or acidulated water, in which are submerged two plates of dissimilar metals, *C* and *Z*, or one metal and a metalloid.

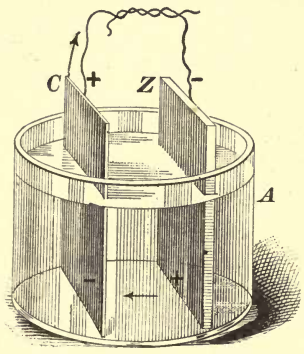

FIG. 52.

Electrolyte is the name given to the liquid that, as it transmits the current, is decomposed by it.

The two dissimilar metals, when spoken of separately, are called *voltaic elements;* when taken collectively, they are known as a *voltaic couple.*

A *voltaic battery* is a number of simple voltaic cells properly joined together.

Electrodes, or *poles*, of a cell or battery are metallic *terminals* attached to the plates, and are used to connect the cell or battery to any exterior conductor or to another cell or battery.

It should be remembered that the polarity of that end of the plate or voltaic element, which is acted upon by the electrolyte, is always of opposite sign to its *electrode.* For instance, in the case of the zinc and copper, the electrode fastened to the zinc would be spoken of as the negative electrode of the cell; while the zinc itself would be the positive element of the cell, its submerged end being positive.

213. Chemical Actions That Take Place in a Simple Cell.—When a piece of ordinary zinc is placed alone in

sulphuric acid diluted with water, the zinc is attacked by the acid and a part of it is dissolved into a salt of that metal called *sulphate of zinc*. At the same time the liquid is decomposed and hydrogen gas is liberated from it, coming up from around the zinc in small bubbles, and the whole mass of the liquid becomes heated. If the zinc is absolutely pure, the chemical actions take place more slowly; the bubbles of hydrogen do not immediately rise to the surface, but form around the zinc, protecting it from further action of the acid. By placing another metal in the water, say a piece of copper, and connecting its exposed end with that of the zinc by a conductor, the chemical actions become exceedingly vigorous again. Large quantities of hydrogen gas are again liberated, but instead of the bubbles appearing around the zinc, they form around the copper and come to the surface at that place; the energy, which in the former case was expended in heating the liquid, now appears in the form of electric energy. Whenever the connection between the exposed ends is broken, all chemical actions cease and remain inactive until the two metals are again connected.

In any voltaic cell, the element that is acted upon by the electrolyte will always be the positive element, and its electrode the negative electrode of the cell.

214. The following list of voltaic elements compose the *electromotive series:*

1. Zinc
2. Cadmium
3. Tin
4. Lead
5. Iron
6. Nickel
7. Bismuth
8. Antimony
9. Copper
10. Silver
11. Gold
12. Platinum
13. Graphite

Any two of these metals form a *voltaic couple*, and produce a difference of potential when submerged in saline or acidulated water, the one standing first on the list being the *positive* element, or plate, and the other, the *negative*. For example, if *nickel* and *graphite* are used, the *nickel* will be acted upon

by the liquid and will form the *positive* element; but if *nickel* and *zinc* are used, the *zinc* will be acted upon by the liquid, and will form the *positive* element.

The difference of potential will be greater in proportion to distance between the positions of the two substances in the list. For example, the difference of potential developed between zinc and graphite is much greater than that developed between zinc and nickel; in fact, the difference of potential developed between zinc and graphite is equal to the difference of potential developed between zinc and nickel, plus that developed between nickel and graphite.

215. Electricity flowing as a current differs from static charges in three important degrees: its *potential* is much lower, its *actual quantity* is larger, and it is *continuous*.

A strong voltaic battery of several cells produces only a slight effect upon a gold-leaf electroscope, and, apparently, none of its parts possess the property of attracting light substances. The potential of a current of electricity is comparatively so small that a voltaic battery composed of a large number of cells is not sufficient to produce a spark of more than $\frac{1}{100}$ or $\frac{1}{200}$ inch in air, whereas a small electrostatic machine will produce sparks several inches in length. If, however, the actual quantity of electricity is measured by its effects in decomposing water, then the quantity produced by a simple voltaic cell as small as a thimble would give greater results than that from an electrostatic machine with plates 2 or 3 feet in diameter.

An electric current cannot be developed upon the surfaces of *non-conducting* substances by current electricity, as in the case of static charges, and it will never flow unless the conducting path is made entirely of *conducting* material.

216. A number of contacts of dissimilar metals can be so arranged as to add their electrical effects together; the difference of potential then developed will be greater in proportion to the number of contacts. Such an arrangement is called a *voltaic pile*. (See Fig. 53.) It is made by

placing a pair of disks of zinc and copper in contact with each other, and then laying a piece of flannel or blotting paper, moistened with brine, upon the copper disk. The pair of disks now form a voltaic couple. Several voltaic couples are placed together and each pair separated by a moistened piece of flannel or blotting paper. One end of such a pile would then be terminated by a disk of copper and the other by a disk of zinc. The copper forms the positive electrode and the zinc the negative electrode. When these two electrodes are joined by a conductor, a current will flow *from* the positive *to* the negative through the conductor, and *from* the negative *to* the positive through the contacts.

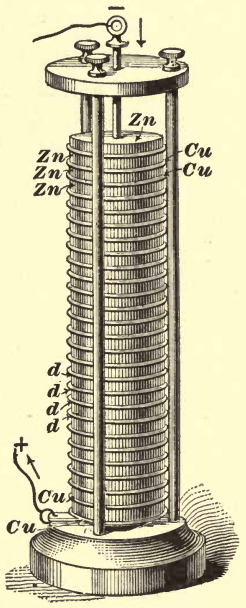

FIG. 53.

217. The difference of potential developed by the mere contact of two dissimilar metals varies, not only with the metals employed and the physical condition of each, but also with their temperature.

The greater difference of potential developed by heat can be shown by soldering one end of a bar of copper to one end of a bar of zinc, and applying heat to the juncture so as to raise its temperature above that of the other parts of the bars. By joining the free ends together with a conductor, a current of electricity will flow *from* the zinc through the contact *to* the copper; then *from* the free end of the copper *to* the free end of the zinc through the conductor. If the junction be cooled below the other parts of the bars, a current is produced in the opposite direction; that is, *from* the copper through the contact *to* the zinc, etc. Even the same metal in different physical conditions will develop a difference of potential if heated in a certain place. For instance, take a copper wire, part of which is straight and

the remainder bent into a spiral, and heat the place where the spiral begins. Under these conditions, a difference of electrical potential will be developed between the two free ends.

In general, the difference of potential is larger in proportion as the difference of temperature increases. With extreme temperature, however, this condition changes, and at a certain temperature of the junction no difference of potential whatever is noticed. This temperature is called the *neutral temperature.* When the junction is heated beyond the neutral temperature, inversion takes place; that is, the direction of the current changes.

Electric currents produced by a change of temperature are called *thermoelectric currents.*

On account of the small difference of potential of thermoelectric currents, they are of no practical value; in fact, they become a source of great annoyance and error in accurate measurements with delicate instruments.

ELECTROMAGNETISM.

218. If a conductor conveying a current of electricity is brought near a freely suspended magnetic needle, the needle tends to place itself at right angles to the conductor, as shown in Fig. 54; or, in general, an electric current and a

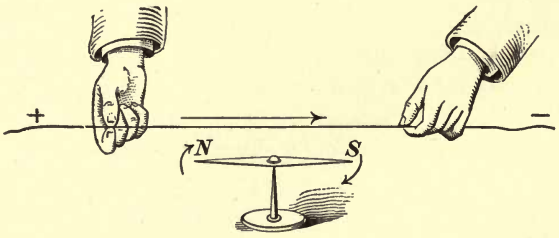

FIG. 54.

magnet exert mutual force upon each other. From the definition given in Art. **174,** the space surrounding the

conductor is a magnetic field. If the conductor is threaded up through a piece of cardboard, and iron filings are sprinkled on the cardboard, they will arrange themselves in concentric circles around the conductor, as represented in Fig. 55. This effect will be observed throughout the entire length of the conductor, and is caused entirely by the current. In fact, every conductor conveying a current of electricity can be imagined as completely surrounded by a sort of magnetic *whirl*, the magnetic density decreasing as the distance from the current increases (see Fig. 56).

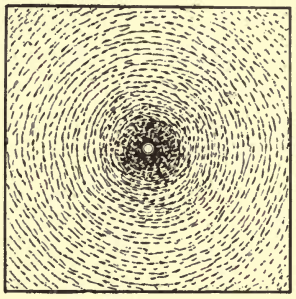

FIG. 55.

219. If the current in a horizontal conductor is flowing

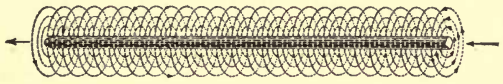

FIG. 56.

towards the north and a compass is placed *under* the conductor, Fig. 57, the north pole of the needle will be deflected towards the *west;* by placing the compass *over* the wire,

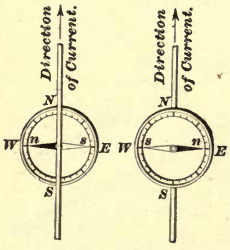

FIG. 57. FIG. 58.

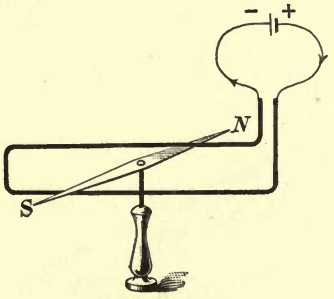

FIG. 59.

Fig. 58, the north pole of the needle will be deflected towards the *east*. By reversing the direction of the current in

the conductor, the needle will point in the opposite direction in each case, respectively.

If the conductor is placed *over* the needle and then bent back *under* it, forming a loop, as shown in Fig. 59, the tendency of the current in both top and bottom portions of the wire is to deflect the north pole of the needle in the same direction.

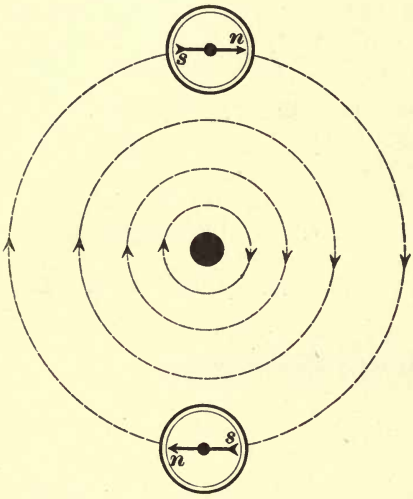

FIG. 60.

From these experiments, knowing the direction of current in the conductor, the following rule is deduced for the direction of the lines of force around the conductor:

Rule.—*If the current is flowing in the conductor away from the observer, then the direction of the lines of force will be around the conductor in the direction of the hands of a watch.*

The direction of the lines of force around a conductor is indicated in Fig. 60, where the current is assumed to be flowing downwards; that is, piercing the paper.

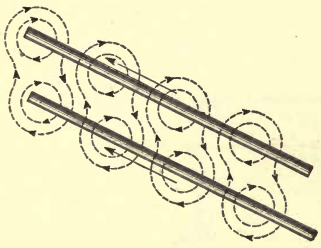

FIG. 61

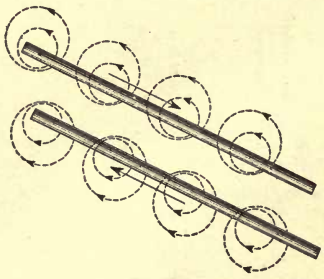

FIG. 62.

220. Two parallel conductors, both transmitting currents of electricity, are either mutually attractive or repellent,

depending upon the relative direction of their currents. If the currents are flowing in the *same* direction in both conductors, as represented in Fig. 61, the lines of force will tend to surround both conductors and contract, thus *attracting* the conductors. If, however, the currents are flowing in opposite directions, as in Fig. 62, the lines of force lying between the conductors will have the same direction and therefore *repel* the conductors.

221. If the conductor carrying the current is bent into the form of a loop, as in Fig. 63, then all the lines of force around the conductor will thread through the loop in the same direction. By bending the conductor into a long *helix* of several loops, the lines of force around each loop will coincide with those around the adjacent loops, forming several long lines of force, which thread through the entire helix, entering at one end and passing out at the other. The same conditions now exist in the helix as exist in a bar magnet; i. e., the lines of force *pass out* from one end and *enter* the other.

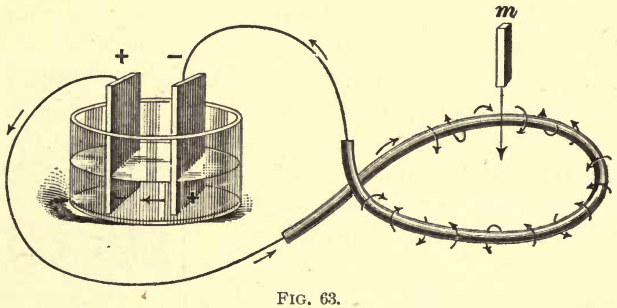

FIG. 63.

In fact, the helix possesses a *north* and a *south pole*, a *neutral line*, and all the properties of attraction and repulsion of a magnet. If it is suspended in a horizontal position and free to turn, it will come to rest pointing in a north-and-south direction. A helix made in this manner, around which a current of electricity is circulating, is called a *solenoid*.

The *polarity* of a solenoid—that is, the direction of the

lines of force that thread through it—depends upon the direction in which the conductor is coiled and the direction of the current in the conductor.

To determine the polarity of a solenoid, knowing the direction of the current :

Rule.—*In looking at the end of the helix, if it is so wound that the current circulates around the helix in the direction of the hands of a watch, that end will be a south pole; if in the other direction it will be a north pole.*

Fig. 64 represents a conductor coiled in a right-handed helix. If the current starts to flow from the end where the observer stands, that end will be a south pole and the observer will be looking through the helix *in the direction of the lines of force.*

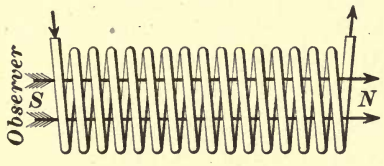

FIG. 64.

The polarity of a solenoid can be changed by reversing the direction of the current in the conductor.

222. In Art. **176** it was stated that when a magnetic substance is brought into a magnetic field, the lines of force in that field crowd together and all try to pass through that substance; in fact, they will alter their circular shape and extend a considerable distance from their original position in order to pass through it. A magnetic substance, therefore, offers a better path for the lines of force than air or other non-magnetic substances.

The facility afforded by any substance to the passage through it of lines of force is called *magnetic permeability*, or simply, *permeability*.

The permeability of all non-magnetic substances, such as air, copper, wood, etc., is taken as 1, or unity. The permeability of soft iron may be as high as 2,000 times that of air. If, therefore, a piece of soft iron is inserted into the magnetic circuit of a solenoid, the number of lines of force will be greatly increased and the iron will become highly magnetized.

223. The Electromagnet.—A magnet produced by inserting a magnetic substance into the magnetic circuit of a solenoid is an *electromagnet*, and a magnetic substance around which the current circulates is called the *core* (see Fig. 65). The solenoid is generally termed the *magnetizing coil.*

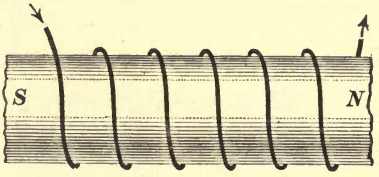

FIG. 65.

In the ordinary form of electromagnet, the magnetizing coil consists of a large number of turns of *insulated* wire, that is, wire covered with a layer or coating of some non-conducting or insulating material, usually silk or cotton; otherwise, the current would take a shorter and easier circuit from one coil to the adjacent one, or from the first to the last coil through the iron core without circulating around the magnet.

224. The simplest form of an electromagnet is the bar magnet. As usually constructed, it consists of a straight bar of iron or steel fitted into a *spool* or *bobbin* made of hard, vulcanized rubber or some other inflexible insulating material. The magnetizing coil of fine, insulated copper wire is wound in layers in the bobbin as shown by the cross-section in Fig. 66.

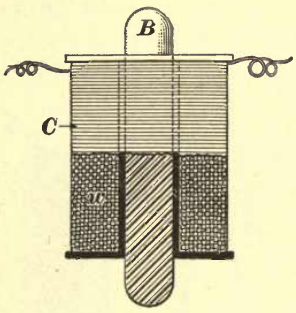

FIG. 66.

The rule for determining the polarity of a solenoid, Art. **221,** is the same for an electromagnet. It makes no difference whether the wire is wound in one layer or in any number of layers, or whether it is wound towards one end and then wound back again over the previous layer towards the other end; so long as the current circulates continually in the same direction around the core, the polarity of the magnet will remain unchanged.

225. The most convenient form of an electromagnet for a great variety of uses is the *horseshoe* or **U**-*shaped* electromagnet. It consists of a bar of iron bent into the shape of a horseshoe with straight ends, and is provided with two magnetizing coils, one on each end of the magnet. The two ends that are surrounded by the coils are the *cores* of the magnet, and the arc-shaped piece of iron joining them together is known as the *yoke* of the magnet. The ordinary **U**-shaped electromagnet is made in three parts; namely, two iron cores wound with the magnetizing coils, and a straight bar of iron joining the two cores together for a yoke, as shown in Fig. 67. In looking at the free ends of the two cores, Fig. 68, the current should circulate around one core in an opposite direction to that around the other. If the current circulates around both cores in the same direction, the lines of force produced in the two cores oppose each other, respectively, forming two like poles at their free ends and a *consequent pole* in the yoke. The total number of lines of force produced by both coils will be

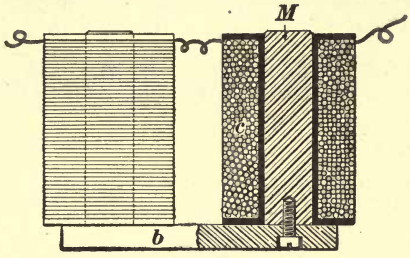

FIG. 67.

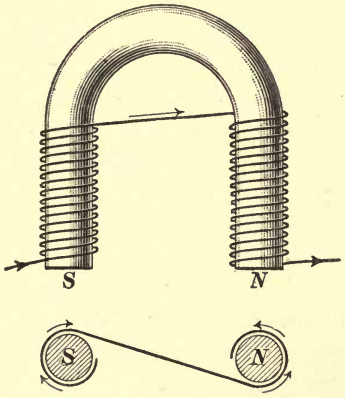

FIG. 68.

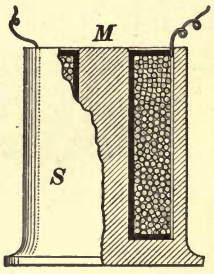

FIG. 69.

greatly diminished and the magnet will exhibit only a small amount of magnetic attraction.

226. Another common form of electromagnet is known as the *iron-clad* electromagnet. In its simplest form, Fig. 69, it contains only one magnetizing coil and one core. The core is fastened to a disk-shaped iron yoke, and the magnetic circuit is completed through an iron shell that rises up from the yoke and completely surrounds and protects the coil.

THEORETICAL CHEMISTRY.

INTRODUCTORY.

MATTER AND FORCE.

1. Matter.—Although the objects we see around us differ one from another to a great extent in appearance, shape, and character, they all possess one property in common, and that is **weight.** All objects are attracted by the earth, and the reason why we find a body to be heavy is that the attraction of the earth, known as *gravitation*, offers a certain resistance, which is commonly termed *weight*, to any efforts made to raise the body from the surface of the earth. Weight is the characteristic property of all substances that can be handled and examined. It is convenient to have a collective name for all such bodies, and accordingly the term *matter* is used.

Matter, then, may be defined as "anything that possesses weight; that is, anything that is acted on by gravitation."

2. Physical State of Matter.—According to the temperature, matter exists in one or another of three distinct physical states, called, respectively, the *solid*, the *liquid*, and the *gaseous* state.

A body in the **solid** state is one whose molecules change their relative position with great difficulty; as, iron, wood, silver, steel, etc.

A body in the **liquid** state is one whose molecules change their relative position with ease. Liquid bodies adapt themselves readily to the shape of the vessels that contain them, and their surface always tends to become perfectly level. Water, sulphuric acid, mercury, etc. are liquids.

A body in the **gaseous** state is one whose molecules tend to separate from one another; as, air, oxygen, hydrogen, etc.

Bodies in the gaseous state are subdivided into two classes: *permanent gases* and *vapors*.

A **permanent gas** is one that remains a gas at ordinary temperatures and pressures.

A **vapor** is a body that at ordinary temperatures is a liquid or solid, but that, when heat is applied to it, becomes a gas, as steam, for instance.

One body may exist in all three states; for example, mercury, which at ordinary temperatures is a liquid, becomes a solid (freezes) at $-40°$ C. and a vapor at 350° C. There is good reason to believe that, at a sufficiently high temperature, all substances would be gaseous, and that at a sufficiently low temperature all would be solid. The point of temperature at which a solid becomes a liquid is called its *melting point;* and that point at which, under ordinary pressure, a liquid becomes a gas or vapor is called its *boiling point.* The number of units of heat required to convert a unit mass of a solid into a liquid is called its *heat of fusion*, or *liquefaction;* and the number of units of heat required to convert a unit mass of a liquid into a vapor, its *heat of vaporization.* The fraction of a unit of heat required to raise the temperature of a unit mass of any substance from 0° C. to 1° C. is called its *specific heat.*

3. Division of Matter.—Science assumes three divisions of matter: *masses*, *molecules*, and *atoms*.

A **mass** of matter is any portion of matter appreciable by the senses.

A **molecule** is the smallest particle of matter into which a body can be divided by physical means; it is the smallest particle that is capable of separate existence.

An **atom** is the still smaller particle produced by the division of a molecule, and is regarded by chemists as the unit quantity of chemical combination.

4. Force.—It would appear that the definition of matter given in Art. **1** is sufficiently extensive to embrace everything, but a little deliberation will show that there are other things besides matter. To give an example, you know that a hammer consists of matter because it is heavy, that is, because it possesses weight; but if with this hammer you strike a number of blows on a piece of iron, you have delivered something that certainly is not matter. The hammer has not lost weight, nor has the iron gained any, but still these blows must be something, as they produce certain effects. For one thing, the piece of iron is altered in shape, it is flattened, and, further, it has acquired temporary heat. Again, to mention another example, if the hammerhead is carefully weighed, and then made red-hot in a fire, it will weigh exactly the same as it did when cold. Further, if it is allowed to cool, this hot piece of steel naturally imparts heat to its surroundings, but still remains unaltered in weight. Here, then, we have something very definite that a body can receive and again yield, and that certainly is not matter. Let us see what relation this something has to matter. In the first example, the blows were delivered by the moving hammer, which consisted of matter in motion. The more rapid the motion, the more violent the blow; in fact, the force of a blow depends both on the quantity of matter and on the rapidity of its motion. Further, the hammered hot iron and the heated hammerhead differ from the same substances in their natural state in that their component particles are in a state of motion (see Art. **5**, *Physics*); when these substances cool, their particles once more enter into a condition of comparative rest. This something, then, that differs from matter is closely related to motion, and is known as *force*. **Force** is, then, *that which is capable of setting matter in motion, or of changing the velocity or direction of the motion of matter.*

The motion of bodies may be divided into three classes: *first*, the motion of the body as a whole, as seen in the case of the moving hammer; *second*, the internal movement of the particles of a body, when it becomes hot; and *third*, atomic movement, which has, however, not yet been fully investigated.

5. Object of Chemistry.—We are now in a position to explain the objects of chemistry. Matter is not only most varied in form, but its form is also continually changing; it is the office of the chemist to investigate these changes, and also the nature of the substances that participate in them. In short, *chemistry is that science which treats of the composition of matter, of the changes produced therein by heat and other natural forces, and of the action and reaction of different kinds of matter on each other.*

6. Difference Between Chemical and Physical Changes.—It is of great importance, even at this early stage, in the study of chemistry, to be able to distinguish between what are simply alterations in the physical properties of matter and what are chemical changes. A description of the following simple experiments will greatly facilitate this object:

If we heat a piece of platinum wire in the flame of a Bunsen burner, we find that it soon becomes white hot; we remove it and allow the wire to cool, and notice that it recovers its original brightness; the heat has not altered it in the least, not even tarnished it.

Next we heat in the same way a piece of bright iron wire; this is tarnished, but has otherwise not undergone any remarkable change.

Now we place a piece of magnesium wire or ribbon in the flame; we notice at once a remarkable action, the wire burns with a peculiar, dazzling white light, depositing a white, easily powdered body, composed of oxygen and magnesium, known as *magnesia*.

What have we learned from these experiments? In the first case, the change produced is merely physical; the hot wire possesses properties very different from the properties

it had when cold, but as soon as it cools again it regains its original character. With the iron wire a slight, permanent change is produced on the surface, which is of a chemical character, but the main portion of the wire is unchanged. In the third experiment there has been a chemical action; the magnesium, as such, has entirely disappeared and a new substance has taken its place.

In short, we have learned that chemical changes are changes that take place within the molecule; that they alter the character of the molecule, and thus cause a change in the identity of the substance itself.

7. Physical and Chemical Properties.—*Physical* properties may be described as those properties that a body possesses as a result of its molecular conditions; while *chemical* properties are those that a body possesses as the result of the atomic composition of its molecule.

Tenacity, which signifies the amount of cohesive attraction, and color, which is the result of the action of the molecules of a body upon light, are properly termed physical properties, while such properties as combustibility, explosibility, affinity, etc. are chemical properties.

8. Mechanical Mixture and Chemical Combination.—It is important to further distinguish between a mere mechanical mixture and a chemical combination; this distinction is most easily made plain by the study of an easy, typical experiment.

If powdered sulphur and fine copper filings are mixed together, the characteristic color of the sulphur as well as that of the copper will disappear, and, to the unaided eye, the mixture will show a uniform greenish color. By the aid of a microscope, however, the particles of copper and sulphur may be distinctly seen lying side by side; we can also easily separate the mixture by washing away the lighter particles of sulphur, and at the same time restore their original colors. Evidently no chemical combination could have taken place; nothing had occurred but a merely *mechanical mixture*. If, however, we heat this mixture in a test tube,

a remarkable change will soon take place; the mixture will begin to glow, and on examining the mass we notice that both the copper and the sulphur have indeed disappeared. We can no longer distinguish either, even with the most powerful microscope, and in their place we find a black substance with properties, such as form, appearance, specific gravity, etc., that are essentially different from those possessed by either copper or sulphur. Here a chemical change has actually occurred. The copper and the sulphur under the influence of heat have combined, and a new substance with different properties has resulted.

These two experiments show conclusively that a chemical combination is widely different from a mere mixture. In the first case, the resultant body is a mean in appearance, specific gravity, etc. between its constituents; each still has its individual properties. In the second case, the resultant body is more or less different in appearance and properties from those of which it is composed. Further, as will be demonstrated later, when substances combine chemically they invariably do so in definite proportions. In a mixture, substances may evidently be present in any proportion whatever.

9. Chemical combination not only produces bodies differing in properties from their constituents, but it also invariably produces *heat*. A striking example is the heat produced in "slaking" lime (that is, in adding water to quicklime); the heat produced is entirely due to the chemical combination of water and quicklime.

Hence, in addition to the difference in appearance, the evolution of heat may be described as the principal characteristic feature by which chemical combination between different substances may be detected.

10. Indestructibility of Matter.—Although chemical action, such as combination, certainly can produce marvelous changes, there is one thing that cannot be accomplished: it can neither create nor destroy matter. By the most careful examination of all known cases of chemical action, it has been positively proved that a loss of matter *never* occurs, that

matter is *indestructible*, and that, even in such chemical actions as the burning and slow disappearing of a candle, there is simply a change of state and not an annihilation of matter. The truth of this great principle in chemical science has been experimentally demonstrated by showing that the weight of substances acting chemically on one another are always the same after chemical changes and actions have taken place as they were before.

11. Synthesis and Analysis.—The composition of chemical compounds is ascertained by two different methods, viz:

1. By *analysis*, which consists in decomposing the compound into its constituent elements.

2. By *synthesis*, which, conversely, consists in uniting the constituents and thus forming the compound.

12. Relation of Chemical Action to Force.—If some red mercuric oxide is placed in a dry and clean test tube and heated, it will be noticed that after a little while the color of the mercuric oxide changes from red to black and that the oxide gradually disappears, while minute globules of mercury condense on the upper part of the tube; in short, the heat has decomposed the mercuric oxide into its constituents, mercury and oxygen.

It has been stated in Art. **9** that when chemical combination occurs heat is produced; in order to effect the change in the experiment described above, heat was used, but it disappeared as heat, being stored up in the constituents in the form of the *available* or *potential energy* of chemical attraction or affinity. If the mercury and oxygen are again combined, the act of recombination will set free the exact amount of heat originally necessary to decompose the compound. Many instances are known in which bodies that evolve a certain heat on uniting may be again decomposed by applying a more intense heat to them. Thus, the following rule has been established: *The quantity of heat evolved during chemical combination is exactly equal to the quantity of heat required to effect the decomposition of the resultant body.*

Further, the chemical action of bodies entering into

combination may be made to produce electricity ; and electricity, again, is able to decompose the resultant bodies. There is also in these cases a definite relation between the amount of chemical action and the quantity of electricity produced.

As we have defined force as that which is capable of setting matter in motion, we must logically rank chemical combination among the forces; for such a typical chemical action as the combustion of coal, for instance, generates heat, and thus indirectly, as in the steam engine, constitutes our most valuable means of setting matter in motion.

13. Relation Between Gravitation, Cohesion, and Chemical Attraction.—As we have seen in Art. **4**, *Physics*, matter is maintained in its normal state by the action of *gravitation*, *cohesion*, and *chemical attraction*. Gravitation acts through all space, and exerts an attraction between all matter. Cohesion acts only across non-appreciable distances, holding together the particles of solids, and also, though to a less extent, those of liquids.

Cohesion may very often be overcome by heat; for example, ice may be converted into water, and a further increase in the temperature will destroy the still prevailing cohesion and will produce steam, which, in common with all other gases, is devoid of cohesion between the particles. But whether ice, water, or steam, the substance remains, so far as its chemical properties are concerned, exactly the same. If, however, we apply a still more intense heat, the steam will be decomposed into oxygen and hydrogen. Hence, heat not only overcomes cohesion, but also destroys the affinity, or chemical attraction, that binds the atoms of substances together.

WEIGHTS AND MEASURES.

METRIC SYSTEM OF WEIGHTS AND MEASURES.

14. Instead of employing the rather complicated English system of weights and measures, most chemists substitute the very simple and convenient metric system.

The metric system is based on the meter, which, according to the United States Coast and Geodetic Survey Report of 1884, is equal to 39.370432 inches. The value commonly used is 39.37 inches, and is so authorized by the United States Government.

There are three principal units: the meter, the liter (pronounced *lee-ter*), and the gram—the unit of length, capacity, and weight, respectively. Multiples of these units are obtained by prefixing to the names of the principal units the Greek words *deka* (10), *hekto* (100), and *kilo* (1,000); the submultiples, or subdivisions, are obtained by prefixing the Latin words *deci* ($\frac{1}{10}$), *centi* ($\frac{1}{100}$), and *milli* ($\frac{1}{1000}$). These prefixes form the key to the entire system. In the following tables, the abbreviations of the principal units of these submultiples begin with a small letter, while those of the multiples begin with a capital letter. The abbreviations are those adopted by the International Congress of Metric Weights and Measures. Chemists commonly use c. c. instead of cm.^{3} for cubic centimeter.

TABLE 1.

MEASURES OF LENGTH.

10 millimeters (mm.)	=	1 centimeter	cm.
10 centimeters	=	1 decimeter	dm.
10 decimeters	=	1 meter	m.
10 meters	=	1 dekameter	Dm.
10 dekameters	=	1 hektometer	Hm.
10 hektometers	=	1 kilometer	Km.

MEASURES OF SURFACE (NOT LAND).

100 square millimeters (mm^2.)	=	1 square centimeter	cm^2.
100 square centimeters	=	1 square decimeter	dm^2.
100 square decimeters	=	1 square meter	m^2.

MEASURES OF VOLUME.

1,000 cubic millimeters (mm^3.)	=	1 cubic centimeter	cm^3.
1,000 cubic centimeters	=	1 cubic decimeter	dm^3.
1,000 cubic decimeters	=	1 cubic meter	m^3.

MEASURES OF CAPACITY.

10 milliliters (ml.)	=	1 centiliter	cl.
10 centiliters	=	1 deciliter	dl.
10 deciliters	=	1 liter	l.
10 liters	=	1 dekaliter	Dl.
10 dekaliters	=	1 hektoliter	Hl.
10 hektoliters	=	1 kiloliter	Kl.

NOTE.—The liter is equal to the volume occupied by 1 cubic decimeter of pure distilled water at its temperature of maximum density, or 4° C.

MEASURES OF WEIGHT.

10 milligrams (mg.)	=	1 centigram	cg.
10 centigrams	=	1 decigram	dg.
10 decigrams	=	1 gram	g.
10 grams	=	1 dekagram	Dg.
10 dekagrams	=	1 hektogram	Hg.
10 hektograms	=	1 kilogram	Kg.
1,000 kilograms	=	1 ton	T.

NOTE.—The gram is the weight of 1 cubic centimeter of pure distilled water at a temperature of 4° C.; the kilogram is the weight of 1 liter of water; the ton is the weight of 1 cubic meter of water.

TABLE 2.

CONVERSION TABLE: ENGLISH MEASURES INTO METRIC.

English.	Metric. Inches to Meters.	Metric. Feet to Meters.	Metric. Pounds to Kilograms.	Metric. Gallons to Liters.
1	0.0253998	0.3047973	0.4535925	3.7853122
2	0.0507996	0.6095946	0.9071850	7.5706244
3	0.0761993	0.9143919	1.3607775	11.3559366
4	0.1015991	1.2191892	1.8143700	15.1412488
5	0.1269989	1.5239865	2.2679625	18.9265610
6	0.1523987	1.8287838	2.7215550	22.7118732
7	0.1777984	2.1335811	3.1751475	26.4971854
8	0.2031982	2.4383784	3.6287400	30.2824976
9	0.2285980	2.7431757	4.0823325	34.0678098
10	0.2539978	3.0479730	4.5359250	37.8531225

TABLE III.

CONVERSION TABLE: METRIC MEASURES INTO ENGLISH.

Metric.	English. Meters to Inches.	English. Meters to Feet.	English. Kilograms to Pounds.	English. Liters to Gallons.
1	39.370432	3.2808693	2.2046223	0.2641790
2	78.740864	6.5617386	4.4092447	0.5283580
3	118.111296	9.8426079	6.6138670	0.7925371
4	157.481728	13.1234772	8.8184894	1.0567161
5	196.852160	16.4043465	11.0231117	1.3208951
6	236.222592	19.6852158	13.2277340	1.5850741
7	275.593024	22.9660851	15.4323564	1.8492531
8	314.963456	26.2469544	17.6369787	2.1134322
9	354.333888	29.5278237	19.8416011	2.3776112
10	393.704320	32.8086930	22.0462234	2.6417902

15. By using these tables, one measure can be converted into another by simple addition. It should be remarked that it is not necessary to use all the figures of the different decimals, four decimal places being all that are commonly used; it is only in very exact calculations that all seven decimal places are required.

ILLUSTRATION.—Change 6,471.8 feet into meters. Any number, as 6,471.8, may be regarded as $6,000 + 400 + 70 + 1 + .8$; also $6,000 = 1,000 \times 6$; $400 = 100 \times 4$, etc. Hence, looking in the left-hand column of Table 2 for figure 6 (the first figure of the given number) we find opposite in the third column, which is headed "Feet to Meters," the number 1.8287838. Now, using but four decimal places and increasing the figure in the fourth decimal place by 1 (since the next figure is greater than 5), we get 1.8288. In other words, 6 feet = 1.8288 meters; hence, simply moving the decimal point three places to the right, 6,000 feet $= 1,000 \times 1.8288 = 1,828.8$. In like manner, 400 feet = 121.92 meters; 70 feet = 21.336 meters; 1 foot = .3048 meter;

and .8 foot = .2438 meter. Adding, as shown below, the result is 1,972.6046 meters.

1828.8000
121.9200
21.3360
.3048
.2438

1972.6046

Again, convert 19.635 Kg. into pounds. The work should be perfectly clear from the explanation given below. The result is 43.2877 pounds.

22.0462
19.8416
1.3228
.0661
.0110

43.2877

The only difficulty in applying these tables lies in locating the decimal point. Its position may always be determined by applying the following statement: If the figure considered lies to the left of the decimal point, count each figure in order, beginning with units (but calling units place zero) until the desired figure is reached; then move the decimal point to the *right* as many places as the figure being considered is to the left of the decimal point. Thus, in the number 6,471.8 above, the 6 lies three places to the left of the 1, which is in units place; hence, the decimal point is moved three places to the *right*. By exchanging the words "right" and "left," the statement will also apply to decimals. Thus, in the second case above, the 5 of the number 19.635 lies three places to the *right* of units place; hence, the decimal point in the number taken from the table is moved three places to the *left*.

ENGLISH WEIGHTS AND MEASURES.

16. Familiarity with English weights and measures is assumed; still, the following particulars may prove useful:

One United States gallon has a volume of 231 cubic inches, and contains 4 quarts, or 8 pints. A gallon of pure water weighs 128 ounces, or 58,318 grains; hence, a pint of pure

water weighs 16 ounces, or 1 pound, and from this has come the saying "a pint is a pound," although, of course, a pint of any liquid having a specific gravity differing from that of water would have a different weight. The measure termed a *fluid ounce* is a measure of volume, and not of weight, and is equal to $\frac{1}{16}$ part of a pint. In other words, it is the volume of 1 ounce of pure water.

The fluid ounce bears the same relation to the avoirdupois ounce as does the cubic centimeter to the gram. When directions are given to dissolve 1 part by weight of a solid in 10 parts (or any other number of parts) by weight of water, either ounces in fluid ounces or grams in cubic centimeters may be taken.

As it is the general custom to use the centigrade thermometer in all chemical calculations and investigations, the temperatures given in this and the following Instruction Papers will always be in degrees centigrade, unless otherwise stated.

The student should bear in mind that the absolute temperature of a body is its temperature in degrees C. $+ 273\frac{1}{3}°$, but in all practical work the fraction is dropped and 273° is used.

THE BALANCE.

17. Though the student has, at this early stage, no need of a balance and a set of weights, a full knowledge of how to manipulate the former is of such vast importance to chemists that a full description will not be out of place at this point.

A most useful form of balance, having a weighing capacity of from 1 milligram to 500 grams, is shown in Fig. 1. The balance is enclosed in a glass case with sliding doors, and rests on a solid wooden box *A A*, in which is a drawer that may be used for weights, etc. A metallic pillar *B* is screwed to the top of the box, and supports, in a cradle *C* at the top, the central knife edges of the beam *D*. On the ends of the beam, at *E*, are also fitted knife edges, on which hang the hooks *F*, and from these are suspended the bows *G* which

carry the pans *H*. In the center of the front of the base of the balances is fixed a handle, or knob *I*. In the position shown in the figure, the pans are at rest on the balance base. But, on turning this knob to the right, the whole of the balance is raised by means of an eccentric, which actuates a sliding rod within the pillar *B*, and raises the cradle *C*. The

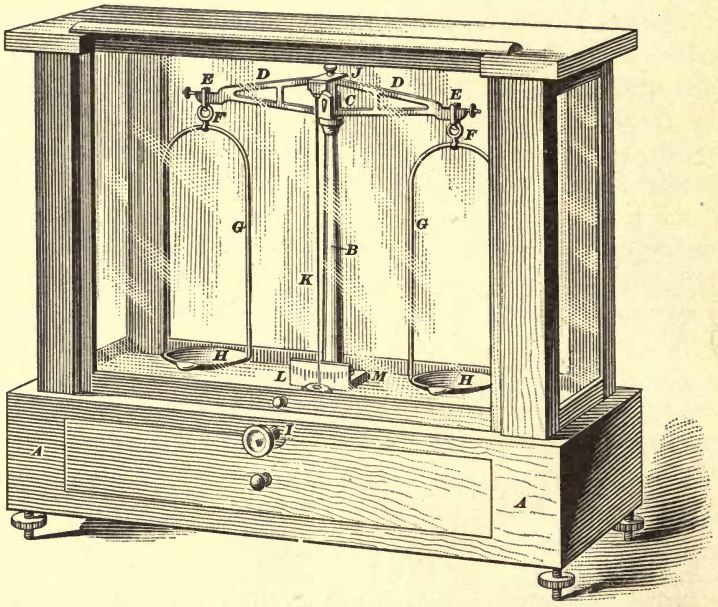

FIG. 1.

pans are then free to swing, and their motion is shown by the index finger *K*, which is fastened to the center of the beam. The end of this index moves in front of a graduated scale, which has a zero mark in the center. Before using the balance, see that it is level and that the index is exactly in front of the *0*-mark. On raising the pans by turning the handle *I*, the index should vibrate an equal number of degrees each side of the zero. Should it not do so, one of the pans is slightly heavier than the other, probably through not being quite clean; if so, turn the handle back and set the balance to rest and then dust the whole instrument very

carefully, and test again. If still out, the balance may require adjusting, which is done by means of a screw at the end of the beam. If the left-hand pan appears to be too heavy, the screw must be screwed outwards from the center of the beam, and in the opposite direction if the right pan is too heavy. However, do not be too quick in adjusting; try repeatedly to get the scales to balance before touching the adjustment screw; many a good balance has been spoiled and accurate weighing has been made impossible by too frequent adjustment.

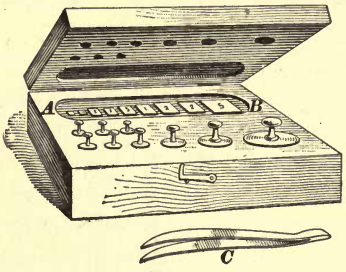

FIG. 2.

Every chemist should keep his balance and weights sacred, and should treat them with the greatest care and consideration.

The weights are usually kept in a box, as shown in Fig. 2. The larger weights are generally made of brass, while the smaller ones are made of aluminum; each weight rests in a separate compartment under a glass lid *A B*. For the purpose of lifting the

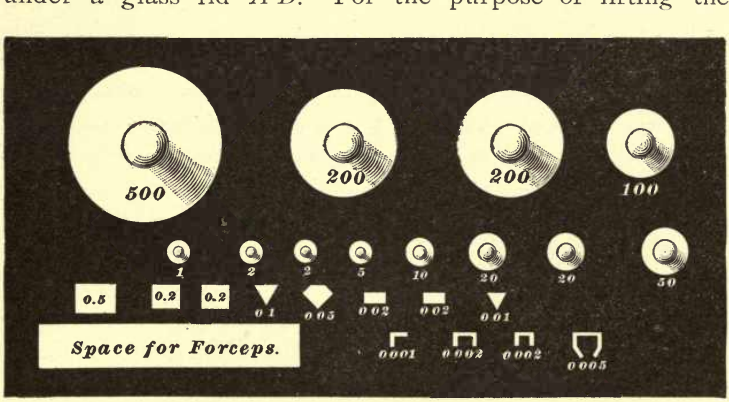

FIG. 3.

weights, a forceps *C* is used, which has its place in the box.

Accurate weights must under no consideration be touched with the fingers, but always lifted with the forceps.

The arrangement of the weights varies with different makes. Fig. 3 shows the usual and most convenient arrangement for a set of weights of from 500 grams to .0001 gram.

Before attempting to weigh, the student must learn not only the denomination of each weight, but also its place in the box. *He must be quite as well able to read the weights he has placed in the balance pan from the empty space in the box as from the weights themselves.*

18. Method of Weighing.—Let us suppose that it becomes necessary to weigh a glass flask. This, in the first place, must be thoroughly cleaned and dried, then placed on the left-hand pan of the balance. Let us assume the weight of the flask to be 238.847 grams and let us see how these figures are obtained. First, with the balance at rest, place the apparatus in position. Then take the 200-gram weight from the box by means of the forceps and place it on the right-hand pan; lift the balance by turning the knob *I* to the right, and notice which pan is heavier; lower the balance by returning the knob. In this case the weight will not be heavy enough, and the index will swing to the right. Put in addition the 100-gram weight in the pan and lift the balance—the weight is too heavy and the index will swing to the left. Do not forget that every time a weight (even the smallest) is added or removed, the balance must be set to the position of rest; i. e., lowered by turning the handle to the left, and that this must be done very gently and carefully. Take off the 100-gram weight and try 50—too much; remove the 50 and try the 20—not enough; add another 20—too much; remove the 20 and add 10—not enough; add 5—not enough; add 2—not enough; add another 2—too much; remove the 2 and add 1—not enough. As the weights get nearer and nearer to the weight of the flask the beam begins to swing more and more slowly. Having now got to a point at which another gram more than turns the scale, commence in the same way with the fractions of a gram. Try first .5 gram—not enough; add .2—not enough;

add another .2—too much; remove the .2 and try .1—not enough. As at this stage the beam swings very slowly, it is advisable to start it by wafting a current of air down on one of the pans with the hand, until the index finger swings very nearly to the extremity of the scale. Try next the .05-gram weight; the index finger swings perhaps just 6 divisions to the left and 5 to the right—the weight is too much; remove the .05 and try .02—not enough; add another .02—not enough. Next add .005—not enough; but more likely than not the index finger swings only a fraction of a degree farther to the right than to the left. Add .002—the index finger swings the same number of degrees each side of zero. The weights now exactly balance the flask. Now we must read the weights; this must first be done from the box, reading the empty spaces. In this case we have $200+20+10+5+2+1 = 238$. Against the words "weight of the flask" write this number in your note book. Next read off the decigram weight spaces; there are empty $.5+.2+.1 = .8$. Write "8" after 238. The centigrams come next; there are $.02+.02 = .04$. Write "4" after 238.8. The milligrams are read in the same manner, and are $.005+.002 = .007$. Write "7" after 238.84. The whole figure will then read, "Weight of flask = 238.847 grams."

Having thus read the weight from the empty spaces in the box, next take out the weights and check off your reading as the weights are returned to their proper places. This double reading greatly reduces the chances of error in recording the weight of a substance. This operation has been described in full, because it is the foundation of exact chemical work; these operations are much shorter in practice than they appear on paper, and the student should in no way become discouraged.

Remember in every case to use the greatest care when working with a balance; never touch scales or weights with the fingers, nor leave them about out of their proper place in the box; on the contrary, take every precaution to see that they are not soiled or otherwise injured.

LAWS OF CHEMICAL COMBINATION.

MOLECULES AND ATOMS.

19. Chemical Definition of Molecule.—A molecule is a group of two or more atoms that are united by their affinity. It is the smallest part of any substance that can be obtained by physical division and still exist in a free or uncombined state.

20. Classification of Molecules. — Molecules are divided into two classes:

1. *Elemental* molecules, which are formed by like atoms.
2. *Compound* molecules, which are formed by unlike atoms.

Matter composed of molecules containing like atoms is termed *simple*, or *elementary*, *matter;* matter whose molecules are composed of dissimilar atoms is called *compound matter.*

ILLUSTRATION.—A mass of gold, of silver, of charcoal, of sulphur, etc., is formed of molecules whose atoms are alike; therefore, these substances are simple, or elementary, matter. The molecules that compose a mass of marble, of sulphuric acid, or of water are composed of unlike atoms; these substances are examples of compound matter.

21. The simplest way to distinguish elemental molecules from compound molecules is to cause a rearrangement of the atoms between two similar molecules. Elemental molecules do not yield any new kind of matter, whereas compound molecules produce elemental molecules.

Let us, for instance, assume that $a\ a$ and $a\ a$ are two molecules, each composed of two atoms a and a; then we never can obtain, by any rearrangement, any other molecules than $a\ a$ and $a\ a$; but should the molecules be $a\ b$ and $a\ b$, that is, compound molecules, and each be composed of the dissimilar atoms a and b, then, by a rearrangement of the atoms, the elemental molecules $a\ a$ and $b\ b$ would result.

22. Number of Elemental Molecules.—Although the number of substances we observe around us seems practically unlimited, yet there are comparatively few elemental

molecules. The number of these that has been positively ascertained so far is approximately seventy, and, as every elemental molecule is composed of atoms that are similar to the molecule, it is self-evident that the number of elemental atoms is also approximately seventy.

23. Nomenclature of Elemental Molecules and Atoms.—Elemental molecules and their atoms always possess the same name, which, in some instances, is the one by which these substances are known in every-day life, as, for instance, gold, silver, iron, copper, etc.; in other cases the name is chosen on account of certain striking properties, etc. that the body may exhibit. So, for instance, *chlorine* obtained its name from the Greek name of its color; *cæsium*, which is the Latin name for "sky-blue," from the blue lines which characterize this element's spectra, etc.

24. Avogadro's Law and the Conclusions Therefrom.—The Italian physicist Avogadro, in 1811, and, independently, Ampère, a French chemist, in 1814, as a result of various investigations and experiments, established the following law, which by right of precedence is generally known as the *law of Avogadro.*

Equal volumes of all substances, either elemental or compound, in the gaseous state, at the same temperature and pressure, contain an equal number of molecules.

From this law it obviously follows:

1. That the molecules of all gaseous bodies must be of equal size.

2. That the weight of any molecule—compared with that of a molecule of hydrogen—is proportional to the weight of any given volume—also compared with an equal volume of hydrogen.

If, for instance, 1 liter of chlorine weighs 35.5 times as much as 1 liter of hydrogen, 1 molecule of chlorine must weigh 35.5 times as much as 1 molecule of hydrogen, if the above law is true.

25. If we mix 1 volume of hydrogen and 1 volume of chlorine and expose the mixture to the light, we obtain

2 volumes of hydrochloric acid; and if we assume that 1 volume of these gases contained 500 molecules of hydrogen and chlorine, respectively, we will have 1,000 molecules of the compound. Submitting hydrochloric acid to an analysis, we find that each of its molecules is composed of 1 atom of hydrogen and 1 atom of chlorine; and since the 1,000 molecules of hydrochloric acid were formed from 500 molecules of hydrogen and 500 molecules of chlorine, it is evident that each of these molecules must have furnished 2 atoms.

From this fact we can state that a molecule of hydrogen is composed of 2 atoms.

If, then, we further assume that the weight of an atom of hydrogen is 1, so as to serve as a unit, the weight of a molecule of hydrogen, i. e., its *molecular weight*, will be 2.

26. By **density** of a body is meant its mass or quantity of matter, compared with the mass or quantity of matter of an equal volume of some standard body arbitrarily chosen; as hydrogen is chosen as this standard, we may obtain the molecular weight of any substance, elemental or compound, by multiplying its density by 2.

ILLUSTRATION.—The density of oxygen is 16; that is, a given volume of it weighs 16 times as much as the same volume of hydrogen; its molecular weight must, then, evidently be 32, or 16 times greater than that of hydrogen.

27. Atomicity.—The number of atoms that compose any elemental molecule is obtained by dividing its molecular weight by its atomic weight. We call an elemental molecule *monatomic*, *diatomic*, *triatomic*, *tetratomic*, or *hexatomic* according as its atomicity is 1, 2, 3, 4, or 6.

Most elemental molecules are diatomic, and those elements that are not volatile, and that, consequently, cannot be weighed in the gaseous state, are arbitrarily classed among the diatomic molecules.

ILLUSTRATION.—The molecular weight of chlorine is 71, its atomic weight is 35.5; hence, the chlorine molecule contains 2 atoms, or is said to be *diatomic*. Arsenic has a molecular weight of 300 and an atomic weight of 75; hence, a molecule of arsenic has an atomicity of 4, or is said to be *tetratomic*.

28. Table 4 contains the elemental molecules the atomicity of which has been experimentally determined.

TABLE 4.

ATOMICITY.

Monatomic	Nitrogen
Mercury	Sulphur (above 800°)
Cadmium	Selenium (above 1,200°)
Zinc	Tellurium
Barium	
Iodine (at 1,500°)	*Triatomic*
Bromine (at 1,800°)	Ozone
Diatomic	Selenium (below 800°)
Hydrogen	*Tetratomic*
Oxygen	Phosphorus
Fluorine	Arsenic
Chlorine	
Bromine	*Hexatomic*
Iodine (below 1,000°)	Sulphur (about 500°)

ATOMIC WEIGHT: VALENCE.

29. For chemical purposes an atom may be defined as the *unit quantity of an element that may enter into, or be expelled from, a chemical combination.*

The *atomic*, or *combining*, *weight* is the relative weight of the atom of each element compared with that of hydrogen, which, being the lightest, is taken as unity.

30. Quantity of Combining Power.—Assuming the combining power of an atom of hydrogen to be 1, the combining power of other atoms will be 1, 2, 3, 4, 5, 6, or 7. This means that some atoms possess a combining power equal to that of hydrogen and unite with one atom of it, while other atoms possess a higher combining power and unite with 2, 3, 4, 5, 6, or 7 atoms of hydrogen.

This quantity of combining power is signified by *valence;* it expresses the number of hydrogen atoms an atom can combine with or be exchanged for; since, however, an atom is

not limited to only one combination with another substance, but can form several compounds with it, its valence necessarily varies. When the valence of a substance changes, it usually increases or decreases by 2; so that an atom of an element may have a valence of 2, 4, or 6, or of 1, 3, 5, or 7. Atoms with even valences are termed *artiads*, those with odd valences, *perissads*.

According as their valence is 1, 2, 3, 4, 5, 6, or 7, atoms are called *monads*, *dyads*, *triads*, *tetrads*, *pentads*, *hexads*, or *heptads*, which names are derived from the Greek numerals. When using the adjective term, the Latin numerals are employed, and we say an atom is *univalent*, *bivalent*, *trivalent*, *quadrivalent*, *quinquivalent*, *sextivalent*, or *septivalent*.

31. To understand valence fully, we must remember that atoms combine through their affinity. This affinity is not equally strong in the atoms of the different kinds of elemental matter. Taking the affinity of hydrogen as unity, any other atom whose affinity is completely satisfied by uniting with 1 atom of hydrogen has the same valence. Other atoms exist whose affinity requires for saturation 2, 3, 4, 5, 6, or 7 atoms; hence their valence—which is the same for all similar atoms—is said to be 2, 3, 4, etc. The valence of oxygen is 2, because 1 atom of oxygen requires 2 atoms of hydrogen to satisfy its affinity. An atom of phosphorus demands 5 atoms of hydrogen to become fully saturated; hence its valence is 5.

32. Graphic Description of the Valence of Atoms.— In order to have a tangible idea of valence, we assume that atoms possess a number of bonds, or links. This is graphically

FIG. 4.

expressed by a circle with short lines radiating from it, the number of which indicates the valence. The graphic representation given in Fig. 4 shows the seven classes of atoms. The number of bonds only, not their direction, is significant.

The valences of the different elements are shown in Table 5. Each element that possesses more than one valence is placed under that heading which signifies the most frequently occurring valence of that element.

TABLE 5.

VALENCE OF ATOMS.*

PERISSADS	ARTIADS
Monads	*Dyads*
Hydrogen	Oxygen
Lithium	Magnesium
Rubidium	Zinc
Cæsium	Cadmium
Potassium	Beryllium
Sodium	Mercury
Silver	Copper
Fluorine, I, III	Calcium
Thallium, I, III	Strontium
Chlorine, I, III, V, VII	Barium
Bromine, I, III, V, VII	Cobalt, II, IV
Iodine, I, III, V, VII	Nickel, II, IV
	Sulphur, II, IV, VI
	Selenium, II, IV, VI
Triads	Tellurium, II, IV, VI
Boron	Manganese, II, IV, VI
Aluminum	*Tetrads*
Gallium	Germanium
Indium	Silicon
Yttrium	Thorium
Scandium	Zirconium
Ytterbium	Carbon
Cerium	Titanium, II, IV
Lanthanum	Tin, II, IV
Erbium	Platinum, II, IV
Gold, I, III	Palladium, II, IV
Antimony, III, V	Lead, II, IV
Bismuth, III, V	Iron, II, IV, VI
Nitrogen, III, V	Chromium, II, IV, VI
Phosphorus, III, V	
Arsenic, III, V	*Hexads*
	Tungsten, IV, VI
	Molybdenum, II, IV, VI
	Ruthenium, II, IV, VI
Pentads	Rhodium, II, IV, VI
Columbium	Iridium, II, IV, VI
Tantalum	Osmium, II, IV, VI
Vanadium, III, V	Uranium, II, IV, VI

* The valence of many of the elements has not been determined with certainty, and consequently it is not claimed that this table is absolutely correct. We have aimed to make it as correct, and at the same time as simple, as possible.

SYMBOLS AND FORMULAS.

33. For convenience of description, each element has an abbreviation of its full name, called its *symbol.* These abbreviations consist, where practicable, of the initial letter of the Latin name of the element. As, however, there are about seventy elements, and only twenty-six letters in the alphabet, a large number of the symbols are composed of the initial and another distinctive letter selected from the name. Thus, the three elements carbon, chlorine, and copper (cuprum) all have names commencing with "C"; carbon being the most important has the letter *C* for its symbol, while *Cl* and *Cu* stand for chlorine and copper, respectively.

As all compound bodies are the result of the combination of elements, they may conveniently be expressed symbolically by placing side by side the symbols of the constituent elements. The symbol of a compound is termed a *formula.* Thus, common salt consists of sodium, whose symbol is *Na*, and chlorine, whose symbol is *Cl*; accordingly, *NaCl* is written for its formula.

The *multiplication of atoms* is expressed by placing an Arabic numeral to the right of and below the symbol.

Thus, S_2 stands for 2 atoms of sulphur; H_3 for 3 atoms of hydrogen.

The *multiplication of a molecule* (formula) is expressed either by enclosing the formula in brackets and placing the multiplier outside to the right of and below the formula, or by placing the multiplier to the left of and in front of the formula.

Either $(H_2O)_4$ or $4H_2O$ stands for 4 molecules of water; either $(O_2)_4$ or $4O_2$ stands for 4 molecules of oxygen, each molecule consisting of 2 atoms.

The valence of an atom is expressed by placing a Roman numeral or a corresponding number of dashes to the right of and above it. C^{iv}, or C'''', stands, for instance, for the quadrivalent atom of carbon and O^{ii}, or O'', stands for the bivalent atom of oxygen.

34. Combination in Definite Proportions. — The symbols, however, signify not only the names of elements, but according to the law — *the proportions by weight, according to which bodies combine, are invariable for each combination*—these symbols also indicate the minimum proportion in which the different atoms combine. For instance, *O* not only signifies oxygen, but it also always stands for 16 parts by weight of it; *Cl*, which stands for chlorine, always signifies 35.5 parts by weight. In this way each of the elements has its own symbol, representing the name and the definite proportion, by weight, in which it combines.

From this it is evident that we not only express by a formula the qualitative, but also the quantitative, composition of a substance.

ILLUSTRATION.—H_2SO_4 stands for sulphuric acid, consisting of
H_2 = 2 parts by weight of hydrogen;
S = 32 parts by weight of sulphur;
O_4 = 64 parts by weight of oxygen.

35. Combination in Multiple Proportions.—The combinations of oxygen and hydrogen, H_2O (water) and H_2O_2 (hydrogen peroxide), afford examples of two elements combining in more than one proportion. It will be observed from the two formulas that in the latter compound exactly twice the quantity of oxygen is combined with the same quantity of hydrogen as in the former. Taking, also, other series of compounds into consideration, it will be seen that a simple relation exists between the proportions of each element in the various compounds. In fact, *not only does chemical combination invariably occur in definite proportion, but when two elements combine in more than one proportion they unite in multiple proportions.* The reason why combination occurs in multiple proportion is, that when two elements combine, 1 atom of the first may combine with either 1, 2, or 3 atoms of the second, but combination with the fraction of an

atom would necessarily be a contradiction of the atomic theory.

ILLUSTRATION.—Nitrogen forms five distinct compounds with oxygen; if such quantities of the compound be taken as contain the same weight of nitrogen, the weights of oxygen will be proportional to the numbers 1, 2, 3, 4, and 5.

Hyponitrous oxide N_2O, 28 parts of nitrogen and 16 parts of oxygen.
Nitrogen dioxide N_2O_2, 28 parts of nitrogen and 32 parts of oxygen.
Nitrous oxide N_2O_3, 28 parts of nitrogen and 48 parts of oxygen.
Nitrogen tetroxide N_2O_4, 28 parts of nitrogen and 64 parts of oxygen.
Nitric oxide N_2O_5, 28 parts of nitrogen and 80 parts of oxygen.

THE CHEMICAL ELEMENTS.

36. The names of the elements, their symbols, and their combining weight are given in Table 6. The combining weights given are those derived from experiments. For most calculations the nearest whole number or whole number with .5 may be employed. Thus, chlorine may be taken as 35.5, oxygen as 16, and so on.

37. Metals and Non-Metals.—The elements have been divided into two groups, according to whether they are metallic or non-metallic in their nature. The metallic properties are very decided in a metal such as either iron, copper, or gold; and it is equally obvious that such bodies as sulphur and phosphorus are not metals. But, although the properties of these are very definite, a number of other bodies are much less marked in character; in fact, there is no well defined line of division between the two classes, as the one series gradually overlaps the other. Thus, the element *arsenic*, which occupies an intermediate position, is placed by some chemists among the metals, and by others among the non-metals or *metalloids*.

In physical character the metals are, as a rule, opaque bodies with a peculiar luster, known as metallic luster, and they are comparatively good conductors of heat and electricity. These

properties, however, do not belong exclusively to the metals, for carbon in the form of graphite has, as is well known, a very decided metallic luster, and conducts electricity well.

Chemically, the metals, as a whole, form oxides that act as bases, while the non-metallic oxides form acids, but even in this respect the series overlap each other. (The meaning of the terms *oxide*, *base*, and *acid* is fully explained in subsequent articles.) The non-metals in Table 6 are indicated by being printed in **heavy type,** while the metals are printed in ordinary type.

As hydrogen is the lightest known element, it has been customary to call its atomic weight 1, and state the atomic weights of other elements in terms of hydrogen as 1. That is, the atomic weight of an element has meant that the element was that many times as heavy as hydrogen. As it somewhat simplifies accurate chemical calculations to call the atomic weight of oxygen 16, there is a tendency at present to arbitrarily assign this value to oxygen, and to assign atomic weights to the other elements that will agree with this arbitrarily chosen standard.

In the accompanying table, taken from the atomic weights published early in the year 1899 by the American Chemical Society, weights based on each of these standards are given in separate columns. As it seems more rational to call the lightest element 1, and use this as the measure for the others, this standard has been retained in this and the following sections.

These atomic weights are the results of the latest determinations, and are probably more accurate than any previously published; but as they are not absolutely correct, and will probably be changed from time to time, it is thought best to use the atomic weights most generally used, in this and the succeeding papers, but if the student prefers he can easily substitute the values here given in his later work.

TABLE 6.

NAMES OF ELEMENTS, THEIR SYMBOLS, AND ATOMIC WEIGHTS.

Name.	Symbol.	Atomic Weight.	
		$H = 1.$	$O = 16.$
Aluminum	*Al*	26.91	27.11
Antimony (stibium)	*Sb*	119.52	120.43
Arsenic	*As*	74.44	75.01
Barium	*Ba*	136.39	137.43
Bismuth	*Bi*	206.54	208.11
Boron	*B*	10.86	10.95
Bromine	*Br*	79.34	79.95
Cadmium	*Cd*	111.54	112.38
Cæsium	*Cs*	131.89	132.89
Calcium	*Ca*	39.76	40.07
Carbon	*C*	11.91	12.00
Cerium	*Ce*	138.30	139.35
Chlorine	*Cl*	35.18	35.45
Chromium	*Cr*	51.74	52.14
Cobalt	*Co*	58.55	58.99
Columbium	*Cb*	93.02	93.73
Copper (cuprum)	*Cu*	63.12	63.60
Erbium	*E*	165.06	166.32
Fluorine	*F*	18.91	19.06
Gallium	*G*	69.38	69.91
Germanium	*Ge*	71.93	72.48
Gold (aurum)	*Au*	195.74	197.23
Hydrogen	*H*	1.00	1.008
Indium	*In*	112.99	113.85
Iodine	*I*	125.89	126.85
Iridium	*Ir*	191.66	193.12
Iron (ferrum)	*Fe*	55.60	56.02
Lanthanum	*La*	137.59	138.64
Lead (plumbum)	*Pb*	205.36	206.92
Lithium	*Li*	6.97	7.03
Magnesium	*Mg*	24.10	24.28
Manganese	*Mn*	54.57	54.99

TABLE 6—*Continued.*

Name.	Symbol.	Atomic Weight.	
		$H = 1.$	$O = 16.$
Mercury (hydrargyrum)	*Hg*	198.49	200.00
Molybdenum	*Mo*	95.26	95.99
Nickel	*Ni*	58.24	58.69
Nitrogen	*N*	13.93	14.04
Osmium	*Os*	189.55	190.99
Oxygen	*O*	15.88	16.00
Palladium	*Pd*	105.56	106.36
Phosphorus	*P*	30.79	31.02
Platinum	*Pt*	193.41	194.89
Potassium	*K*	38.82	39.11
Rhodium	*Rh*	102.23	103.01
Rubidium	*Rb*	84.78	85.43
Ruthenium	*Ru*	100.91	101.68
Samarium	*Sm*	149.13	150.26
Scandium	*Sc*	43.78	44.12
Selenium	*Se*	78.58	79.17
Silicon	*Si*	28.18	28.40
Silver (argentum)	*Ag*	107.11	107.92
Sodium (natrium)	*Na*	22.88	23.05
Strontium	*Sr*	86.95	87.61
Sulphur	*S*	31.83	32.07
Tantalum	*Ta*	181.45	182.84
Tellurium	*Te*	126.52	127.49
Thallium	*Tl*	202.61	204.16
Thorium	*Th*	230.87	232.63
Tin (stannum)	*Sn*	118.15	119.05
Titanium	*Ti*	47.79	48.15
Tungsten (wolfram)	*W*	183.43	184.83
Uranium	*U*	237.77	239.59
Vanadium	*V*	50.99	51.38
Ytterbium	*Yb*	171.88	173.19
Yttrium	*Y*	88.35	89.02
Zinc	*Zn*	64.91	65.41
Zirconium	*Zr*	89.72	90.40

At ordinary temperatures, two of the elements—mercury and bromine—are liquids; five—oxygen, hydrogen, chlorine, fluorine, and nitrogen—are gases. The remaining elements are solids; but at varying temperatures all have been liquefied, with the exception of carbon, which so far has only been slightly softened with the highest temperature obtainable.

38. Electropositive and Electronegative Elements. A more scientific, but not generally accepted, division is to distinguish the elements according to the quality of their combining power. This means that they are divided into:

1. *Positive* elements, or those that are attracted to the negative electrode in electrolysis.

2. *Negative* elements, or those that are attracted to the positive electrode.

ILLUSTRATION.—If salt is decomposed through the influence of electricity, it will be noticed that the chlorine atoms gather at the positive electrode, and are, consequently, called negative, while the sodium atoms collect at the negative electrode and are considered positive atoms.

In Table 7 the principal elements are arranged according to their electrochemical character, each atom being positive to any atom placed above it, and negative to any one placed below it.

It should be remarked here, that it is *absolutely necessary* that the student should make himself thoroughly familiar with the names of the elements, their symbols, atomic weights, and electrochemical characters; in fact, he should try to learn these things by heart; nothing looks so bad as to see a chemist referring to his books to find the symbol, combining weight, etc. of an element.

39. Meta-Elements.—In Art. **22** the number of elements is given as "approximately seventy"; the reason for not giving a more positive figure is that the standard adopted for elements has become doubtful through recent researches.

It has, for instance, been observed that the spectrum of

TABLE 7.

ELECTROCHEMICAL CHARACTER OF ELEMENTS.

Negative End
Oxygen
Sulphur
Nitrogen
Fluorine
Chlorine
Bromine
Iodine
Selenium
Phosphorus
Arsenic
Chromium
Vanadium
Molybdenum
Tungsten
Boron
Carbon
Antimony
Tellurium
Tantalum
Columbium
Titanium
Silicon
Tin
Hydrogen
Gold
Osmium
Iridium
Platinum
Rhodium
Ruthenium
Palladium
Mercury
Silver
Copper
Uranium
Bismuth
Gallium
Indium
Germanium
Lead
Cadmium
Thallium
Cobalt
Nickel
Iron
Zinc
Manganese
Lanthanum
Cerium
Thorium
Zirconium
Aluminum
Scandium
Erbium
Yttrium
Ytterbium
Beryllium
Magnesium
Calcium
Strontium
Barium
Lithium
Sodium
Potassium
Rubidium
Cæsium
Positive End

didymium is capable of resolution into nine separate constituents, and that yttrium may be made to yield five different components, each giving a different phosphorescent spectrum. From this the conclusion has been drawn that such elements as didymium, yttrium, samarium, decipium, philippium, holmium, thulium, dyprosium, terbium, gnomium, etc. are not elements in the true sense of the word, but that they are formed of certain simpler substances, which are at present unknown to us, and which are provisionally called *meta-elements*.

40. Determination of Atomic Weights.—The atomic weight has been defined in Art. **29** as the relative weight of any atom referred to hydrogen as unity. In order to determine the atomic weight of an atom, we have to know:

1. The quantity, by weight, of the element that combines with 1 atom of hydrogen.

2. The molecular weight of the hydrogen compound.

As the absolute weight of each constituent of a certain quantity, usually 100 parts, of a hydrogen compound may be obtained by analysis, the quantity, by weight, of a body that unites with 1 part of hydrogen may be obtained by simple proportion, and as the molecular weight of any substance represents the sum of the weights of its constituent atoms, the combining weights obtained by analysis, when added together, must either give the molecular weight, or a number of which the molecular weight is a multiple.

EXAMPLE 1.—A quantity of hydrogen iodide contains, according to analysis, 99.22 per cent. of iodine and .78 per cent. of hydrogen; the molecular weight of hydrogen iodide is 128. What is the atomic weight of iodine?

SOLUTION.—The proportion .78 : 99.22 = 1 : 127 shows that in this compound the smallest quantity of iodine that can combine with 1 part of hydrogen weighs 127 times as much as hydrogen, and as the sum of the combining weights (127 + 1) is 128, which is also the molecular weight of hydrogen iodide, the atomic weight of iodine is 127. Ans.

EXAMPLE 2.—Hydrogen arsenide, or arsine, is shown by analysis to contain 3.85 per cent. of hydrogen and 96.15 per cent. of arsenic; its molecular weight is 78. What is the atomic weight of arsenic?

SOLUTION.—From the proportion $3.85 : 96.15 = 1 : 25$, it is seen that 25 parts of arsenic combine with 1 part of hydrogen. If there is only 1 atom of hydrogen in 1 molecule of arsine, then 25, being the smallest quantity, by weight, in which arsenic combines, will be its atomic weight; but adding the two combining weights $(25 + 1)$ together we obtain only 26, which, however, is exactly one-third of the given molecular weight of arsine. Hence, a molecule of arsine is composed of 3 parts of hydrogen and $25 \times 3 = 75$ parts of arsenic. 75 is, therefore, the atomic weight of arsenic. Ans.

41. Indirect Method for Determination of Atomic Weights.—Some elements do not unite directly with hydrogen; the comparison with hydrogen must, then, be made by means of another element, for which purpose chlorine is generally chosen.

EXAMPLE.—There is no known hydrogen compound of silver, but there exists a well known combination of silver and chlorine, called horn-silver. On analysis, horn-silver yields 75.26 per cent. of silver and 24.74 per cent. of chlorine; its molecular weight is 143.5. What is the atomic weight of silver?

SOLUTION.—As 35.5 parts of chlorine combine with 1 part of hydrogen, the quantity of silver that combines with 35.5 parts of chlorine is its atomic weight; $24.74 : 75.26 = 35.5 : 108$. Since $108 + 35.5 = 143.5$, the molecular weight, 108, thus obtained must be the atomic weight of silver. Ans.

EXAMPLES FOR PRACTICE.

42. Solve the following:

1. The density of arsenic vapor is 150; what is its molecular weight? Ans. 300.

2. The density of phosphorus is 62; what is its molecular weight? Ans. 124.

3. The density of phosphorus is 62, and its atomic weight is 31; how many atoms compose a phosphorus molecule? Ans. 4 atoms.

4. The density of bromine is 80, its atomic weight is 80; how many atoms compose a bromine molecule? Ans. 2 atoms.

5. Water, a compound of hydrogen and oxygen, has a molecular weight of 18. On analysis it yields 88.89 per cent. of oxygen and 11.11 per cent. of hydrogen; what is the atomic weight of oxygen? Ans. 16.

6. Marsh gas is a compound containing 75 per cent. of carbon and 25 per cent. of hydrogen. Its density is 8; what is the atomic weight of carbon? Ans. 12.

7. The analysis of aurous chloride gives 84.7 per cent. of gold and 15.3 per cent. of chlorine. The density of the compound is 116; what is the atomic weight of gold? Ans. 196.5.

8. The analysis of stannous chloride gives 62.43 per cent. of tin and 37.57 per cent. of chlorine. The molecular weight of the compound is 189; what is the atomic weight of tin? Ans. 118.

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43. Relation of Atomic Weight to Specific Heat.— When equal weights of different bodies are raised through the same number of degrees of temperature, they take up different amounts of heat; or, in other words, different bodies possess different capacities for heat. Thus, 1 kilogram of water requires thirty times as much heat as 1 kilogram of mercury to raise its temperature 1 degree; that is, if the quantity of heat required to raise the temperature of 1 kilogram of water 1 degree is represented by 1, the quantity of heat required to raise the same weight of mercury 1 degree will be represented by $\frac{1}{30}$. This fraction expresses what is called the *specific heat* of mercury.

The *specific heat* of a solid or liquid body is, then, the amount of heat required to raise the temperature of a certain weight of this body 1 degree, the amount of heat required to raise the temperature of an equal weight of water 1 degree being taken as unity.

The specific heat of a substance varies according as the substance is in the solid, liquid, or gaseous state, but the specific heat of all solid elements exhibits a remarkable relation to their atomic weight. In 1819 Dulong and Petit observed that if, instead of calculating the specific heats for equal weights, such weights be taken as represent the atomic weights of the solid elements, the numbers expressing the capacity for heat of the different atoms are sensibly equal; from which it follows that all solid elements possess the same *atomic heat*, and that the specific heats of solid elements are inversely as their atomic weights.

Hence, we have in the determination of specific heat a means of checking the atomic weight of a solid element, or of ascertaining it in doubtful cases. This is done by dividing

the atomic heat of a solid element by its specific heat, the quotient thus obtained being the atomic weight. The number 6.4 represents, approximately, the atomic heat of the solid elements.

	Specific Heat.	Atomic Heat.	Atomic Weight.
Illustration.—Lead....	.031	6.41	206.77
Platinum	.032	6.33	197.81
Silver...	.059	6.37	107.97
Sulphur.	.2026	6.48	31.98
Tin.....	.054	6.37	117.96
Copper..	.0951	6.04	63.51

COMPOUND MOLECULES.

44. A compound molecule may be defined as a molecule composed of dissimilar atoms, which are united according to the law of valence. The number of atoms that take part in this union is apparently unlimited; while a molecule of hydrochloric acid contains but two atoms, and a molecule of sulphuric acid 7 atoms, certain organic compounds contain a much higher number. Protagon, for instance, a compound of lecithin and cerebrin, both obtained from the substance of the human brain, contains 384 atoms.

The molecular weight of a compound molecule is equal to the sum of the weights of its component atoms, and also to twice its density in the gaseous state. Thus, an atom of chlorine weighing 35.5 unites with an atom of hydrogen weighing 1, to form a molecule of hydrogen chloride (hydrochloric acid), whose molecular weight is 36.5. Or, 2 atoms of hydrogen weighing 2, 1 atom of sulphur weighing 32, and 4 atoms of oxygen weighing 64 unite to form a sulphuric-acid molecule weighing 98.

45. Compound molecules are classified as:

1. *Binary* molecules; that is, molecules which, whatever the number of atoms may be, contain only two different kinds of atoms.

2. *Ternary* molecules; that is, molecules whose dissimilar atoms are united by the aid of some third atom.

BINARY COMPOUNDS.

46. Nomenclature of Binary Molecules.—The names of chemical compounds are fixed by rule, but in the case of the more well known compounds the old, every-day names are commonly used; hardly any one, for instance, would think of saying "hydrogen monoxide" for water. The name of a compound should, as far as possible, indicate its composition; this end is attained by making the name of a binary compound consist of derivatives of the names of its components. There are, unfortunately, several modifications of each name, and English-speaking chemists have not always used a uniform method of nomenclature; the student, therefore, must make himself familiar with all of them, as, in the course of reading various books and manuals on chemistry, he is sure to meet with the same body under different names. Very little practice is sufficient, however, to overcome this little difficulty.

The names of the binary molecules are derived from their constituent atoms. The most frequently occurring binaries are those composed of a metal and a metalloid (non-metal); the name of the metal is first written, and then that of the metalloid, one or more syllables being removed from the latter and the termination *ide* added.

ILLUSTRATION.—Copper and oxygen yield copper oxide.
Magnesium and oxygen yield magnesium oxide.
Silver and sulphur yield silver sulphide.
Zinc and phosphorus yield zinc phosphide.
Calcium and iodine yield calcium iodide.
Aluminum and bromine yield aluminum bromide.
Sodium and chlorine yield sodium chloride.
Potassium and nitrogen yield potassium nitride.
Barium and fluorine yield barium fluoride.
Cadmium and selenium yield cadmium selenide.

The termination *ide* is always characteristic of a binary compound.

47. Modification of This Rule.—As it frequently happens that more than one compound of the same elements

is known, it becomes necessary to distinguish these compounds.

Whenever the metal atom enters into combination with more than one valence, this fact is indicated in the compound by changing the termination of the name of this atom into *ic* or *ous*.

Should the metal act with only two valences, then in the higher one its name takes the termination *ic*, and in the lower one the termination *ous*.

ILLUSTRATION.—Bivalent mercury and oxygen yield murcuric oxide.
Univalent mercury and oxygen yield mercurous oxide.
Quadrivalent tin and chlorine yield stannic chloride.
Bivalent tin and chlorine yield stannous chloride.

48. Use of Prefixes.—If the metallic constituent acts with more than two valences, the termination *ic*—being given on the discovery of the compound—is generally arbitrarily assigned, and a further discrimination becomes again necessary; this distinction is marked by the use of a prefix. A compound in which the valence of the first, or metallic, constituent is *less* than in the *ous*, takes the prefix *hypo*, which means "under." When the valence is above *ic*, the prefix *per* is used. The termination of the second constituent, however, always remains *ide*.

ILLUSTRATION.—Quinquivalent chlorine and oxygen form chloric oxide.
Trivalent chlorine and oxygen form chlorous oxide.
Univalent chlorine and oxygen form hypochlorous oxide.
Septivalent chlorine and oxygen form perchloric oxide.
Sexivalent sulphur and oxygen form sulphuric oxide.
Quadrivalent sulphur and oxygen form sulphurous oxide.
Bivalent sulphur and oxygen form hyposulphurous oxide.
Quinquivalent nitrogen and oxygen form nitric oxide.
Trivalent nitrogen and oxygen form nitrous oxide.
Univalent nitrogen and oxygen form hyponitrous oxide.

49. Use of Numeral Prefixes.—In order to indicate the number of atoms of each constituent, the Greek numerals are prefixed to the name of the constituent.

ILLUSTRATION.—1 atom of *C* and 2 atoms of *O* form carbon dioxide.
1 atom of *P* and 5 atoms of *Br* form phosphorous pentabromide.
2 atoms of *Fe* and 3 atoms of *O* form diferric trioxide.
3 atoms of *Ti* and 4 atoms of *N* form trititanic tetranitride.

50. Formation of Binaries.—A binary molecule consists of atoms with either equal or different valences. Atoms having the same valence unite in the proportion of 1 to 1; that is, the atoms mutually saturate each other and their chemism is satisfied.

ILLUSTRATION.—K' and Cl', both monads, form $K'Cl'$ or $K{-}Cl$.
S'' and O'', both dyads, form $S''O''$ or $S{=}O$.
Pt^{iv} and Si^{iv}, both tetrads, form $Pt^{iv}Si^{iv}$ or $Pt{\equiv}Si$.

When atoms with different valences unite, each atom has to furnish the same number of bonds to satisfy their chemism. This number is the least common multiple of the two different valences. The absolute number of atoms of each constituent is obtained by dividing the least common multiple by the valence of each atom.

ILLUSTRATION.—When a tetrad and a hexad combine, each must furnish 12 bonds, 12 being the least common multiple of 4 and 6. To do this, 3 tetrads and 2 hexads are required. The following formulas will further illustrate this law:

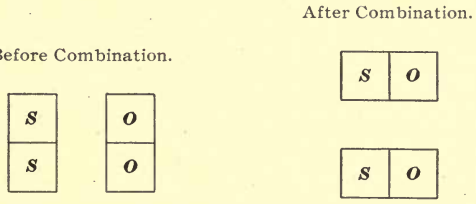
H', a monad, and O'', a dyad, form H'_2O''.
N''', a triad, and O'', a dyad, form $N'''_2O''_3$.
Sn^{iv}, a tetrad, and S'', a dyad, form $Sn^{iv}S''_2$.
I^{v}, a pentad, and O'', a dyad, form $I^{v}_2O''_5$.
Cl^{vii}, a heptad, and O'', a dyad, form $Cl^{vii}_2O''_7$.
Si^{iv}, a tetrad, and F', a monad, form $Si^{iv}F'_4$.

Whenever a perissad and an artiad atom form a molecule, the required number of atoms is inversely as the valence of each.

51. It was previously stated that atoms do not exist in a free state; consequently, the formation of compound

molecules may be considered as a process in which an exchange of atoms takes place between a number of molecules that are brought in close proximity; a fact that may easily be explained by the theory that dissimilar atoms possess a far greater attraction for each other than similar atoms.

ILLUSTRATION.—A molecule of sulphur $S=S$ is brought under suitable conditions in contact with a molecule of oxygen $O=O$. The stronger attraction of the atoms of oxygen for the atoms of sulphur causes them to form a new combination, and the result, in this case, is $SO + SO$, or 2 molecules of hyposulphurous oxide. Graphically, it may be represented in the following way:



52. Saturated and Unsaturated Molecules.—The groups of atoms now being considered are called saturated *molecules*, because the bonds of their constituent atoms are all mutually engaged. We distinguish between these and *unsaturated groups of atoms*, which, possessing free bonds, may enter into combination like single atoms. These unsaturated groups of atoms are known as *compound radicals* (or radicles). They do not exist free in nature, although, like atoms, by combining with another similar group they may form a saturated molecule. Their valence is equal to the number of their unsatisfied bonds, which is the difference of the valence of their components.

ILLUSTRATION.—Water consists of $H\text{–}O\text{–}H$; supposing 1 atom of hydrogen to be removed, the unsaturated group $H\text{–}O\text{–}$ remains, which, though consisting of 2 atoms, is able to enter into the formation of a molecule like any single atom. It possesses one free bond and acts therefore as a monad. It is expressed in formulaic form thus: $(H'O'')'$. Combining with another similar group it forms a saturated molecule $(H'O'')' + (H'O'')' = H\text{–}O\text{–}O\text{–}H$, or H_2O_2. Or the pentad P^v may be partly saturated by the dyad O'', forming the trivalent compound

radical $(P^{v}O'')'''$, which may combine like a trivalent atom, say with Cl, forming $(P^{v}O'')'''Cl_3$.

53. Names of Compound Radicals.—The names of compound radicals terminate in *yl*. The root of the name comes either from constituents of the radical or from some compound into which it enters.

ILLUSTRATION.—The compound radical $(H'O'')'$ is called *hydroxyl*, $(P^{v}O'')'''$ is termed *phosphoryl*, $(C^{iv}H_3)'$ is termed *methyl*, from methyl alcohol, of which it is a constituent. The compound radicals $(H_2N)'$ *amidogen*, $(CN)'$ *cyanogen*, and $(H_4N)'$ *ammonium* are the only three exceptions to this rule.

Molecules containing a compound radical, united to an atom, are regarded as binaries and named accordingly.

ILLUSTRATION.—$(NO_2)'$ and Cl form nitryl chloride, $(CO)''$ and S yield carbonyl sulphide, etc.

TERNARY COMPOUNDS.

54. The name *ternary* is applied to those compounds containing three elements. The third atom, which performs a linking function, must be a polyad, an atom whose valence is greater than 1, since no monad can join other atoms together. Among the compounds higher than the binary series are those formed by the union of water with oxides, thus forming acids and hydrates (hydroxides) and derivatives of these bodies.

55. Acids.—The name acid is a familiar one, because it is continually applied in every-day parlance to anything that is sour. A number of bodies possess this distinction in common; to the chemist the sourness of an acid is but an accidental property, as, according to his definition of these bodies, substances are included as acids that are not sour to the taste. *An acid may be defined as a body containing hydrogen, which hydrogen may be replaced by a metal, or group of elements equivalent to a metal.* It may also be defined as a compound that will unite with a base, forming a salt and water. As a class the acids *are* sour; they are also

active chemical agents. Most acids are characterized by the property of changing the color of a solution of litmus, a naturally blue body, to a red tint.

Oxygen is a constituent of most acids, the members of this group being distinguished as *oxyacids.* The oxides that by union with water form acids are termed *anhydrides*, or *anhydrous acids.* They are in most cases non-metallic oxides, but sometimes consist of metals combined with a comparatively large number of atoms of oxygen. There are a few acids in which oxygen is absent; these are termed *hydracids;* hydrochloric acid *HCl* is an example of this class. According to the definition given, hydrogen is an essential constituent of all acids. It should, however, be mentioned that some chemists apply the term "acid" to what are here termed "anhydrides"; consequently, CO_2 (carbon dioxide or carbonic anhydride) is sometimes described as carbonic-acid gas. This name is now, however, being replaced by the one more in accordance with the theories of modern chemical science.

56. Bases and Alkalies.—The oxides that, when dissolved in water, restore the blue color to reddened litmus are called *alkalies;* they form a subdivision of a larger class of oxides, the whole of which combine readily with acids, and are known as *bases.*

A **base** *is a compound, usually an oxide or a hydrate, of a metal* (*or group of elements equivalent to a metal*), *and this metal* (*or group of elements*) *is capable of replacing the hydrogen of an acid, when the two are placed in contact.* The greater number of metallic oxides are bases. Bases, as well as acids, differ considerably in their chemical activity. As already mentioned, certain bases that dissolve in water are termed *alkalies.*

An **alkali** is a base of a specially active character. It can easily be recognized by its solubility in water, to which it imparts a soapy taste and feeling, and also by its ability to restore the blue color to vegetable blues, such as litmus, that have been previously reddened by an acid. The principal

alkalies are *sodium hydrate* $NaHO$ and *potassium hydrate* KHO. A solution of ammonia gas in water is also alkaline, and is often termed *ammonium hydrate*.

Paper tinted yellow with *turmeric* is also used as a test for alkalies, which give the paper a reddish-brown color.

The *hydrates* are mostly compounds of metallic oxides with water; they are also sometimes termed *hydroxides*. (The word hydroxide, however, is objectionable, because it infringes on the restriction of *ide* to the names of binary compounds.) Their formation is represented by the following equations:

$$\underset{\text{sodium oxide}}{Na_2O} + \underset{\text{water}}{H_2O} = \underset{\text{sodium hydrate}}{2NaHO}$$

$$\underset{\text{calcium oxide}}{CaO} + \underset{\text{water}}{H_2O} = \underset{\text{calcium hydrate}}{Ca(OH)_2}$$

57. Salts.—*When an acid and a base react on each other, the substance or compound produced by the replacement of the hydrogen of the acid by the metal of the base is termed a* **salt.** Water is also formed during the reaction. The action of the stronger acids and bases on each other is very violent; the resulting salts are usually without action on litmus. This is, however, not always the case, for when a strong acid combines with a weak base, the salt is acid to litmus, of which nitrate of mercury is an example. When the base is strong and the acid is weak, as in sodium carbonate, the salt has an alkaline reaction. The following are instances of the formation of salts by the union of acids and bases:

$$\underset{\substack{\text{hydrochloric}\\\text{acid}}}{HCl} + \underset{\substack{\text{sodium}\\\text{hydrate}}}{NaHO} = \underset{\substack{\text{sodium}\\\text{chloride}}}{NaCl} + \underset{\text{water}}{H_2O}$$

$$\underset{\substack{\text{hydrochloric}\\\text{acid}}}{HCl} + \underset{\substack{\text{ammonium}\\\text{hydrate}}}{NH_4HO} = \underset{\substack{\text{ammonium}\\\text{chloride}}}{NH_4Cl} + \underset{\text{water}}{H_2O}$$

$$\underset{\substack{\text{sulphuric}\\\text{acid}}}{H_2SO_4} + \underset{\substack{\text{calcium}\\\text{oxide}}}{CaO} = \underset{\substack{\text{calcium}\\\text{sulphate}}}{CaSO_4} + \underset{\text{water}}{H_2O}$$

Litmus, which has been repeatedly spoken of in connection with the ternary compounds, is itself a salt of a vegetable acid and a vegetable base possessing a blue color. An acid, when added, displaces the weak vegetable acid and with the remaining vegetable base forms a salt; the liberated litmus acid, being red, gives the red tint to the solution. On adding to this solution another base, this latter combines with the stronger acid, displacing in turn the vegetable base. The displaced vegetable acid and base again unite, and the original blue color is also restored.

58. Nomenclature of Acids and Salts.—The names of acids are derived from those of their principal constituents, changing them into adjectives ending in *ic*. From sulphur we have *sulphuric acid;* from nitrogen, *nitric acid;* etc. *Hydracids* are distinguished by the prefix *hydro;* for instance, *hydrochloric acid.*

The names of the corresponding salts are derived from the same root by adding *ate*. Thus, the salts of sulphuric and nitric acids are called *sulphates* and *nitrates*, respectively.

If an element should form *two* oxides, both of which combine with water to form acids, then the acid containing the *higher* proportion of oxygen receives the name ending in *ic*, while for the termination of the other acid *ous* is chosen.

ILLUSTRATION.—Two oxides of sulphur are known, SO_2, or sulphurous oxide, and SO_3, or sulphuric oxide. Each combines with water to form acids.

$$\underset{\text{sulphurous oxide}}{SO_2} + \underset{\text{water}}{H_2O} = \underset{\text{sulphurous acid}}{H_2SO_3}$$

$$\underset{\text{sulphuric oxide}}{SO_3} + \underset{\text{water}}{H_2O} = \underset{\text{sulphuric acid}}{H_2SO_4}$$

The salts of acids whose names end in *ous* have the termination *ite*.

$$\underset{\text{sulphurous acid}}{H_2SO_3} + \underset{\text{sodium hydrate}}{2NaHO} = \underset{\text{sodium sulphite}}{Na_2SO_3} + \underset{\text{water}}{2H_2O}$$

The same termination that is applied to the base is also applied to the salt. Thus, two different sulphates of mercury are obtained, namely, when sulphuric acid is caused to react with mercurous oxide and mercuric oxide, respectively.

$$\underset{\text{sulphuric acid}}{H_2SO_4} + \underset{\text{mercurous oxide}}{Hg_2O} = \underset{\text{mercurous sulphate}}{Hg_2SO_4} + \underset{\text{water}}{H_2O}$$

$$\underset{\text{sulphuric acid}}{H_2SO_4} + \underset{\text{mercuric oxide}}{HgO} = \underset{\text{mercuric sulphate}}{HgSO_4} + \underset{\text{water}}{H_2O}$$

The salts of hydracids, being necessarily binaries, have names ending in *ide*.

$$\underset{\text{hydrochloric acid}}{2HCl} + \underset{\text{mercuric oxide}}{HgO} = \underset{\text{mercuric chloride}}{HgCl_2} + \underset{\text{water}}{H_2O}$$

59. Basicity of Acids.—In the formation of salts, the hydrogen of an acid is replaced by a metal. Hydrogen, being univalent, is replaced by different metals in proportions varying according to their valence. The valence of the displacing metals must be equal to that of the total number of hydrogen atoms displaced. Thus, a monad will replace 1 atom of hydrogen; a dyad, 2 atoms, and so on. The quantity of a base required to form a salt with an acid depends on the number of atoms of hydrogen present capable of replacement by the base; the measure of this quantity is termed *basicity of the acid.* Those acids containing but 1 atom of replaceable hydrogen are termed *monobasic*, those with 2 or 3 atoms of replaceable hydrogen are called *dibasic* and *tribasic*, respectively.

A monobasic acid requires 1 atom of a monad metal to form a salt, while with a bivalent metal 2 molecules of the acid are necessary. In every case such a number of molecules of acid and atoms of metal must be taken as will give the same total valence of replaced hydrogen atoms as that

of the replacing metal. The following typical examples will illustrate this principle:

MONOBASIC ACID.

$$HNO_3 + Na'OH = Na'NO_3 + H_2O$$

nitric acid — sodium hydrate — sodium nitrate — water

$$2HNO_3 + Ca''(OH)_2 = Ca''(NO_3)_2 + 2H_2O$$

nitric acid — calcium hydrate — calcium nitrate — water

$$3HNO_3 + Bi'''(OH)_3 = Bi'''(NO_3)_3 + 3H_2O$$

nitric acid — bismuth hydrate — bismuth nitrate — water

DIBASIC ACID.

$$H_2SO_4 + 2Na'OH = Na_2SO_4 + 2H_2O$$

sulphuric acid — sodium hydrate — sodium sulphate — water

$$H_2SO_4 + Ca''(OH)_2 = CaSO_4 + 2H_2O$$

sulphuric acid — calcium hydrate — calcium sulphate — water

$$3H_2SO_4 + 2Bi'''(OH)_3 = Bi'''_2(SO_4)_3 + 6H_2O$$

sulphuric acid — bismuth hydrate — bismuth sulphate — water

TRIBASIC ACID.

$$H_3PO_4 + 3Na'OH = Na'_3PO_4 + 3H_2O$$

phosphoric acid — sodium hydrate — sodium phosphate — water

$$2H_3PO_4 + 3Ca''(OH)_2 = Ca''_3(PO_4)_2 + 6H_2O$$

phosphoric acid — calcium hydrate — calcium phosphate — water

$$H_3PO_4 + Bi'''(OH)_3 = Bi'''PO_4 + 3H_2O$$

phosphoric acid — bismuth hydrate — bismuth phosphate — water

60. Acid, Normal, and Basic Salts.—Salts such as those whose formulas are given in the preceding examples are known as *normal salts. A* **normal salt** *may be defined as a salt produced by the replacement of the hydrogen of the acid by its valence equivalent of a metal, or by a group of elements that acts as a metal, in a base.*

In the case of those acids containing more than 1 replaceable atom of hydrogen, if a portion only of such hydrogen is

replaced, the result is a salt that still possesses an acid character. *An* **acid salt** *may, then, be defined as a salt produced by the partial replacement of the hydrogen atoms of a polybasic acid by the metal of a base;* as, for instance:

$$\underset{\text{sulphuric acid}}{H_2SO_4} + \underset{\text{sodium hydrate}}{NaHO} = \underset{\text{hydrosodium sulphate}}{NaHSO_4} + \underset{\text{water}}{H_2O}$$

Certain oxyacids exist that possess the power of combining with more of the base than is necessary to produce a normal salt, thereby forming what are termed *basic salts.* *A* **basic salt** *may, then, be defined as a salt produced by the combination of an acid with a higher proportion of a base than is necessary for the formation of a normal salt.* Yellow basic mercuric sulphate, known as turpeth mineral, is an example of a basic salt, the additional quantity of base being distinguished here by being placed after the comma in the formula $HgSO_4,(HgO)_2$. Its formula is generally written thus: $Hg_3(SO_6)$.

61. Classification of Acids.—Oxyacids are sometimes divided into *ortho-* and *meta-acid*s, but this distinction is not important and these terms are often used in a rather loose way. In its strictest sense, an ortho-acid is one containing as many hydroxyl groups as the maximum valence of the principal element. Thus, phosphorus, having a maximum valence of 5 (as is shown by the formation of PCl_5), might be expected to form an acid having the formula $P(OH)_5$, and this would be called *ortho-phosphoric* acid. As a matter of fact, this acid is not known, and only a very few acids formed in this way do exist; hence, the term is often used in a broader sense and is applied to the normal tribasic acids. Thus, $PO(OH)_3$ is frequently called ortho-phosphoric acid.

A *meta-acid* is one formed by extracting water from an *ortho-acid.* Hence, if we use the term *ortho* in its strictest sense in the case just mentioned, and call $P(OH)_5$ ortho-phosphoric acid, the acid usually called ortho-phosphoric $PO(OH)_3$, which would be formed by removing one molecule

of water from $P(OH)_5$, would be a meta-acid; and by removing another molecule of water we would have another meta-acid $PO_2(OH)$, which is usually known as *meta-phosphoric* acid. Thus, we would have two meta-acids of phosphorus. To avoid confusion, when the term *ortho* is used in its strictest sense, the term *anhydro-acids* is sometimes applied to those obtained by the removal of water from the ortho-acids, and the term *meta* is reserved for a particular acid. This distinction is only important in studying the theoretical formation of acids, but the terms *ortho* and *meta* are important in organic chemistry where they are applied in a different manner.

VOLUME RELATIONS OF MOLECULES.

62. Molecular Volume.—The fact that all gases, whether elementary or compound, expand and contract at the same rate when subjected to variations of temperature and pressure, has an important bearing on their probable molecular conditions. Their similarity in this respect has led to the assumption expressed in *Avogadro's law* (see Art. **24**), from which it follows that, at the same temperature and under the same pressure, the volume of any gaseous molecule is the same, whatever may be the nature and composition of the gas. Every molecular formula, therefore, not only expresses the weight but also the volume of the molecule. The molecule of hydrogen is chosen as the standard of molecular volume; but as it is sometimes convenient to speak of atomic volume—which, however, is inconsistent since atoms do not exist free—the volume occupied by an atom of hydrogen is taken as unity; the volume occupied by a molecule of hydrogen will be 2, and, since all molecules occupy the same volume, the volume of a molecule of any gaseous body is assumed to be 2, also, when this system is used.

63. Density and Molecular Weight.—The density of a gas has already been defined as the weight of any volume

compared with that of the same volume of hydrogen (see Art. **26**). As the molecule of hydrogen contains two atoms, its molecular weight, expressed in terms of atomic weight, is, consequently, 2. The molecular weight of any body in the gaseous state is the weight of that volume which occupies the same space as do 2 parts by weight of hydrogen. Conversely, as the molecular weight is the sum of the weights of the constituent atoms, the density of a gas may be determined from its formula. As the molecule of hydrogen weighs 2, the density of any gas, whether elementary or compound, is identical with the half of its molecular weight. For instance, the molecule of steam weighs 18, and occupies the same space as the molecule of hydrogen, which weighs 2; the quantity of steam occupying the same space as 1 volume of hydrogen must, therefore, weigh $\frac{18}{2} = 9$.

64. Absolute Weight of Hydrogen.—Hydrogen being taken as the unit of density, its absolute weight is of great importance. As a result of most careful weighing, it has been found that 1 liter of hydrogen at the normal temperature and pressure weighs, approximately, .0896 gram, or 11.16 liters weigh 1 gram. It is absolutely necessary for the student to remember that the weight of 1 liter of hydrogen at 0° and 760 millimeters pressure is .0896 gram, and he should therefore thoroughly impress it on his mind. Knowing this, there is absolutely no difficulty in calculating the weight of any other gas whose composition is known. *The weight in grams of a liter of any gas is its density multiplied by .0896; further, the weight in grams of 11.16 liters of any gas is identical with the number representing its density.* The density of oxygen, for instance, being 16, the weight of 1 liter at N. T. P.* is $.0896 \times 16 = 1.4336$ grams. The weight of a compound gas is found just as easily. The density of hydrochloric acid is 18.18; therefore, the weight of 1 liter of this acid is $.0896 \times 18.18 = 1.6289$ grams, or 11.16 liters weigh 18.18 grams.

* N. T. P. is used as abbreviation of normal temperature and pressure.

This constant, .0896, is of such frequent occurrence in chemical calculations that it has been proposed to give it a distinct name, *crith;* the weight of a liter of any other gas is then expressed as so many criths.

65. Diffusion of Gases.—Diffusion is the intermixture of molecules brought about by their power of moving against one another. This power is possessed in the highest degree by gaseous molecules, all gases being capable of perfect and comparatively rapid intermixture. Some liquids, such as alcohol and water, also intermix perfectly, although comparatively slowly; while other liquids, such as oil and water, diffuse into each other only to a very limited extent. When alcohol is poured on to water it forms a separate layer on the surface of the water, because it is specifically the lighter of the two. After a time, however, the two layers will no longer be discernible, and the liquid will be a homogeneous mixture of alcohol and water. In the same way, hydrogen will float on air, but only for a very short time, since the rate of diffusion of this gas is extremely rapid. A homogeneous mixture of hydrogen and air will be speedily formed.

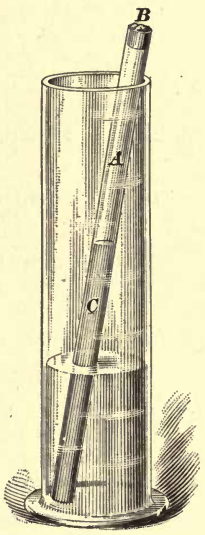


FIG. 5.

The molecules of all gases are constantly moving, but they do not move with the same velocity, so that some gases diffuse more rapidly than others. This has been discovered by confining a gas in a vessel closed by some material having very minute pores, and immersing the vessel in an atmosphere of some other gas. The passage of the molecules through the pores of the material closing the vessel is sufficiently slow to allow of a comparison between the velocities of passage or *rates of diffusion* of the two gases.

For this purpose, a diffusion tube (see Fig. 5) is employed. It consists of a glass tube *A*, closed at one end by a plate of plaster of Paris *B*. If this tube is filled with hydrogen and

its open end immersed in colored water, the water will be observed to rise rapidly in the tube on account of the rapid escape of the hydrogen through the pores of the plaster of Paris. The external air, of course, passes into the tube through the pores at the same time, but much less rapidly than the hydrogen passes out, so that the ascent of the column of water *C* marks the difference between the volume of hydrogen that passes out and the volume of air that passes into the tube in a given time, and allows a measurement to be made of the rate of diffusion, that is, of the velocity with which the gas issues as compared with the velocity with which the air enters, this velocity being always taken as unity. To determine the rate of diffusion it is, of course, necessary to maintain the water at the same level within and without the diffusion tube, so as to exclude the influence of pressure.

This method has disclosed the *law of diffusion*, known as *Graham's law*, namely, that *the rate of diffusion of gases is inversely as the square root of their density.* Thus, hydrogen and oxygen have, respectively, densities of 1 and 16; therefore, hydrogen diffuses four times as rapidly as oxygen.

66. Combination by Volume.—Gay-Lussac, who investigated the proportions in which gaseous volumes enter into combination, established the following laws:

1. *The ratio in which gases combine by volume is always a simple one.*

2. *The volume of the resulting gaseous product bears a simple ratio to the volume of its constituents.*

The laws of combination by volume, which at the time of Gay-Lussac's investigation (between 1805 and 1808) were simply experimental, have recently been investigated by Clausius, who has shown that they are really nothing but a very simple deduction from the law of Avogadro.

According to Avogadro's law, equal volumes of all gaseous bodies contain the same number of molecules. If, therefore, the number of molecules is in some way decreased, the

volume must necessarily be decreased also. If we assume that in a given volume of any gas each molecule is diatomic, and that by some means the molecule can be made tetratomic —the number of atoms remaining the same—the number of molecules will be reduced one-half, since each molecule contains twice the number of atoms; and this reduction of molecules is naturally accompanied by a corresponding decrease of the volume; that is, the volume of the gas will also be reduced one-half.

Or, if we assume that the diatomic molecule could be made triatomic, the number of molecules, and, consequently, the volume, would be reduced one-third.

To apply this argument to the facts of volume combination, we will consider separately the combination of hydrogen with monads, dyads, triads, and tetrads, assuming all their molecules to be diatomic.

1. *Combination of a Monad with Hydrogen.*—Each atom of a monad combines with an atom of hydrogen; before they combine, their molecules are each diatomic, and each molecule of the compound is also diatomic; in accordance with the principles of Art. **24,** if there was 1 volume of each before the combination, the number of compound molecules after the combination takes place is exactly equal to the sum of the molecules of the monad and hydrogen before they combined, and the result is 2 volumes of the compound.

Monads, then, combine with one another, volume to volume, yielding 2 volumes of the product.

Illustration.—The monad atom chlorine *Cl* and the monad atom hydrogen *H* combine to form hydrogen chloride *HCl*. Their molecules being diatomic, they unite, molecule to molecule, volume to volume, and, consequently, the number of molecules and the volume they occupy will remain exactly the same after combination has taken place as it was previous to it.

We may represent this molecularly thus:

$$\begin{array}{ccccc} H & & Cl & & H-Cl \\ | & + & | & \text{yield} & \\ H & & Cl & & H-Cl, \end{array}$$

and volumetrically thus:

H_2 and Cl_2 form $(HCl)_2$

2. *Combination of a Dyad with Hydrogen.*—A dyad atom will require 2 atoms of hydrogen; 1 molecule will require 2 molecules of hydrogen; and 1 volume, 2 volumes of hydrogen. Or, to express this fact in other words, dyads combine with monads in the ratio of 1 to 2.

As the resulting molecule is triatomic, it is evident that 3 diatomic molecules have combined to form 2 triatomic molecules, and that the decrease in the number of molecules also causes a corresponding decrease in volume. Thus we see that 3 volumes of simple gas give 2 volumes of compound gas, a condensation of 3 volumes to 2 volumes taking place during combination.

Dyads, then, combine with monads in the ratio of 1 to 2, and yield 2 volumes of the product.

ILLUSTRATION.—An atom of oxygen is bivalent, and its molecule diatomic. One atom of oxygen requires 2 atoms of hydrogen to form a molecule of water gas, and 1 molecule of oxygen requires 2 molecules of hydrogen, and 1 volume requires 2 volumes. This may be represented molecularly thus:

Before combination.	After combination.
$H\ O\ H$	$H-O-H$
$\vert\ \vert\vert\ \vert$	$H-O-H$
$H\ O\ H$	

3. *Combination of a Triad with Hydrogen.*—One triad atom requires 3 monad atoms for saturation, 1 molecule requires 3 molecules, and 1 volume requires 3 volumes; since the newly produced molecule is tetratomic, the number of molecules and the corresponding volume they occupy must be reduced one-half.

Triads, then, unite with monads in the ratio of 1 to 3, yielding 2 volumes of the product.

ILLUSTRATION.—The trivalent atom of nitrogen N requires 3 atoms of hydrogen to form 1 molecule of ammonia.

$$\begin{array}{ccccccc} N & & HHH & & HHH & & HHH \\ \vert\vert\vert & + & \vert\ \vert\ \vert & = & \diagdown\vert\diagup & + & \diagdown\vert\diagup \\ N & & HHH & & N & & N \end{array}$$

or $\boxed{H_2} + \boxed{H_2} + \boxed{H_2} + \boxed{N_2} = \boxed{(H_3N)_2}$

As 4 diatomic molecules yield 2 tetratomic molecules, 1 volume of nitrogen and 3 volumes of hydrogen will yield 2 volumes of ammonia gas.

4. *Combination of a Tetrad with Hydrogen.*—One tetrad atom requires 4 monad atoms for saturation; 1 tetrad molecule will require 4 monad molecules, and 1 volume of any tetrad will require 4 volumes of any monad; as the resulting molecule consists of 5 atoms, the 5 original volumes will be condensed to two.

Tetrads, then, unite with monads in the ratio of 1 to 4 and yield 2 volumes of the product.

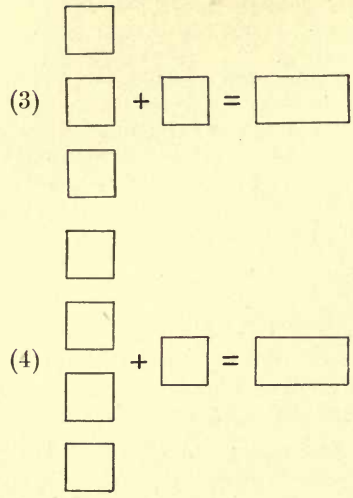
ILLUSTRATION.—The tetrad atom of carbon *C* combines with 4 monad atoms of hydrogen to form marsh gas. That is, 5 diatomic molecules unite to form 2 pentatomic molecules, or, as in this case, 1 volume of carbon and 4 volumes of hydrogen yield 2 volumes of marsh gas.

$$\begin{array}{ccccccc} C & & HHHH & & H & & H \\ \vert\vert\vert\vert & + & \vert\ \vert\ \vert\ \vert & = & \vert & & \vert \\ C & & HHHH & & H-C-H & + & H-C-H \\ & & & & \vert & & \vert \\ & & & & H & & H \end{array}$$

The preceding combinations may be graphically represented thus:

(1) $\square + \square = \square$

(2) $\begin{matrix}\square\\ \square\end{matrix} + \square = \square$



(1) 1 monad molecule and 1 monad molecule yield 2 diatomic molecules.

(2) 1 dyad molecule and 2 monad molecules yield 2 triatomic molecules.

(3) 1 triad molecule and 3 monad molecules yield 2 tetratomic molecules.

(4) 1 tetrad molecule and 4 monad molecules yield 2 pentatomic molecules.

67. There are two more cases to be considered, namely, where the molecules are either monatomic or tetratomic.

1. As all known monatomic molecules are dyads, the combination with monad atoms would be:

1 atom and 2 atoms give 3 atoms;
1 molecule and 1 molecule give 1 molecule;
1 volume and 1 volume give 1 volume;

or, in other words, monatomic dyad molecules combine with diatomic monad molecules in equal volume and yield 1 volume of the product.

ILLUSTRATION.—The monatomic dyad zinc *Zn* combines with the diatomic monad chlorine *Cl*; that is, 1 atom of zinc combines with

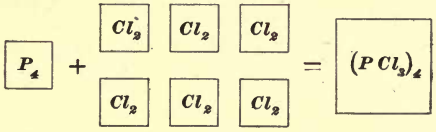
2 atoms of chlorine, or 1 molecule of zinc combines with 1 molecule of chlorine, or 1 volume of zinc with 1 volume of chlorine. The product is the triatomic molecule zinc chloride $ZnCl_2$, a condensation of half the original volume taking place.

2. As all known tetratomic molecules are triads, the atomic combination is:

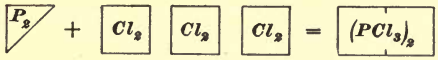
1 atom and 3 atoms yield 4 atoms;
1 tetratomic molecule and 6 diatomic molecules yield 4 tetratomic molecules;
1 volume and 6 volumes yield 4 volumes;

which means that tetratomic triad molecules combine with diatomic monad molecules in the ratio of 1 to 6 volumes, producing 4 volumes of the product.

ILLUSTRATION.—Phosphorus is a triad, possessing a tetratomic molecule. When it unites with chlorine, we have, atomically, P and Cl_3 giving PCl_3, and molecularly, P_4 and $6Cl_2$ giving $(PCl_3)_4$, or by volume:



or, as it is also sometimes graphically expressed,



CHEMICAL CALCULATIONS.

REACTIONS.

68. Molecular Stability.—All molecules are more or less liable to chemical change. Their atoms may be altered in number, in kind, or in relative position by various external influences. A molecule is the more stable in proportion as it resists this tendency to change.

69. Chemical Reactions and Reagents.—The term *chemical reaction* is applied to any change that takes place in the atoms composing a molecule. The substance applied to produce the change is called a *reagent.*

70. Chemical reaction always occurs within the molecule. Hence, when two substances react on each other, the changes that result may be considered as taking place between single molecules; and as all molecules in homogeneous matter are alike, and what is true of one molecule must be true for any mass of them, it follows that a molecular change represents a mass change, also.

71. Chemical Equations. — Since every chemical change or reaction is simply an alteration in the association and position of atoms within the molecule, and, consequently, a change of the constitution of the molecule, this reaction may be expressed by an equation.

The substances entering into the reaction are called the *factors;* those issuing from the reaction are called the *products.*

The equation representing the reaction is written according to the following rule:

Place the formulas of the factors—connected by the sign of plus—as the first member of the equation, and the formulas of the products—also connected by the sign of plus—as the second.

ILLUSTRATION.—The reaction of the 2 molecules $A\,B$ and $C\,D$ would then be expressed as follows:

$$A\,B + C\,D = A\,D + B\,C.$$

72. A chemical equation, however, not only expresses the fact of reaction, but it also indicates the quantities, by weight concerned in it, and since the atoms remain exactly the same after the reaction as they were before, being only differently associated, it is evident that a loss of weight cannot occur through chemical reaction, and the weight of the bodies resulting from a chemical change must be the

same as that of the bodies before the change, whatever it may be, had occurred.

73. Chemists classify chemical reactions in the following manner:

Analytical reactions represent the separation of a complex molecule into more simple ones.

ILLUSTRATION.— $2H_2O = 2H_2 + O_2$
water hydrogen oxygen

Synthetical reactions represent the union of simple molecules to form one or more complex ones.

ILLUSTRATION.— $H_2 + Cl_2 = 2HCl$
hydrogen chlorine hydrochloric acid

Metathetical reactions represent an exchange or transposition of atoms between molecules.

ILLUSTRATION.— $Pb(NO_3)_2 + Na_2SO_4 = Pb(SO_4) + 2NaNO_3$
lead nitrate sodium sulphate lead sulphate sodium nitrate

74. Facility of Chemical Reaction.—Since a chemical reaction is the result of the reciprocal action of atoms and has for its effect a change in the composition of the molecule, it is evident that it can only take place when these atoms, and, consequently, their molecules, are brought into intimate relation, or, more exactly, when the molecules of one of the bodies enter within the sphere of action of other bodies. This sphere is rather limited, as the affinity of the atoms is only exercised at infinitely small distances. In consequence, affinity is often retarded by cohesion, which maintains the relations between molecules of solid bodies. These forces are frequently in opposition, and in order that the first may obtain supremacy it is necessary that the other shall yield. To increase the affinity between two or more bodies it becomes necessary to diminish their cohesion. On this condition the molecules can enter within the sphere of their reciprocal attraction, and the atoms of one body can attract those of the other.

Experience teaches us that this cohesion can to a certain extent be overcome and chemical reaction facilitated when the acting substances are brought into the liquid or gaseous state. Hence, either heat by which bodies are vaporized, or solution, by which bodies are liquefied, is an important aid in producing chemical reaction.

75. Precipitation.—If we place some solution of mercuric chloride in a test tube and add to it, drop by drop, an iodide of potassium solution, a red powder is formed, which gradually settles down to the bottom of the tube. This powder is mercuric iodide, while potassium chloride remains in solution.

Whenever a substance separates from a solution through the addition of another, as the mercuric iodide has done in the example cited, the term *precipitate* is applied to the separated substance, and the substance that will produce a precipitate is known as a *precipitant*. The process of producing a precipitate is known as *precipitation*.

Those conditions of chemical change depending upon solubility are stated in *Berthollet's law*, which may be expressed as follows:

When, on mixing two substances in solution, a new compound can be formed that is insoluble in the solvent employed, such compound will be formed and will appear as a precipitate.

76. Another law, also established by Berthollet, relates to the products of reactions that are volatile instead of being insoluble solids; it may be stated as follows:

When, on mixing different substances, a new substance that is volatile can be produced by the rearrangement of the atoms of the partaking substances, such new substance will be produced and will appear as a gas.

ILLUSTRATION.—If we mix sodium nitrate $NaNO_3$ and sulphuric acid H_2SO_4 together, through a rearrangement of the atoms of these two

substances, hydrosodium sulphate and nitric acid are produced, according to the equation:

$$NaNO_3 + H_2SO_4 = HNO_3 + HNaSO_4$$

sodium nitrate — sulphuric acid — nitric acid — hydrosodium sulphate

Nitric acid, being volatile, escapes as a gas when heat is applied.

By means of Table 8, which shows the solubility of various compounds, the resulting chemical change of these compounds, covered by the first-mentioned law of Berthollet, may be predicted, and as the student progresses in his study and experimenting he will soon acquire sufficient knowledge to predict also the changes covered by the second-mentioned law of Berthollet, which requires some familiarity with the volatility of the various compounds.

ILLUSTRATION.—The reactions obtained by mixing calcium chloride and sodium carbonate may be expressed by the following equation:

$$CaCl_2 + Na_2CO_3 = CaCO_3 + 2NaCl$$

calcium chloride — sodium carbonate — calcium carbonate — sodium chloride

Will there be a precipitate?

Consulting Table 8, we find that calcium carbonate is insoluble, and we can predict that under the given conditions calcium carbonate will be obtained in the solid form as a precipitate.

The reactions obtained by mixing calcium hydrate and carbon dioxide may be expressed by the equation:

$$Ca(OH)_2 + CO_2 = CaCO_3 + H_2O$$

calcium hydrate — carbonic dioxide — calcium carbonate — water

Will there be a precipitate?

Consulting the table, we find that calcium carbonate is insoluble in water, and, therefore, we can predict that under the given conditions calcium carbonate will be obtained as a precipitate.

77. Modes of Chemical Action.—Although chemical actions vary considerably in their character, they can all be classified under the following five heads:

1. *Direct Union or Synthesis.*—

EXAMPLE.— $Zn + Cl_2 = ZnCl_2$

zinc — chlorine — zinc chloride

TABLE S.

SOLUBILITIES OF CHEMICAL COMPOUNDS.

	Platinic.	Auric.	Mercuric.	Mercurous.	Plumbic.	Arsenic.	Antimonic.	Stannic.	Stannous.	Silver.	Bismuth.	Cupric.	Cadmium.	Ferric.
Hydrate	*A*	*A*	*A*	*A*	*A*	*DW*	*A-I*	*A-I*	*A*	*A*	*A*	*A*	*A*	*A*
Nitrate	*W*	*W*	*W*	*W*	*W*	—	—	*W*	—	*W*	*W*	*W*	*W*	*W*
Carbonate	*A*	—	*A*	*A*	*A*	—	—	*A*	*A*	*A*	*A*	*A*	*A*	*A*
Acetate	—	—	*W*	*DW*	*W*	—	—	*W*	*W*	*DW*	*W*	*W*	*W*	*W*
Oxalate	*W*	—	*A*	*A*	*A*	—	*A*	*W*	*A*	*A*	*A*	*A*	*A*	*A*
Cyanide	*DA*	*W*	*W*	*W*	*A*	—	*A*	—	—	*DA*	*A*	*A*	*DW*	*W*
Chloride	*W*	*W*	*W*	*A*	*DW*	*W*	*W*	*W*	*W*	*I*	*W*	*W*	*W*	*W*
Bromide	*W*	*W*	*DW*	*A*	*DW*	*W*	*W*	*W*	*W*	*I*	*W*	*W*	*W*	*W*
Iodide	*A*	*A*	*DA*	*A*	*DW*	*DW*	*A*	*DW*	*DW*	*I*	*A*	—	*W*	*W*
Fluoride	*W*	*W*	*W*	*A*	*A*	*W*	*W*	*DW*	*W*	*W*	*W*	*DW*	*DW*	*W*
Sulphide	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*
Sulphite	*W*	—	*A*	*A*	*A*	—	*A*	*W*	*DW*	*A*	*A*	*W*	*DW*	*DW*
Sulphate	*W*	*W*	*W*	*DW*	*A-I*	—	*W*	*W*	*W*	*DW*	*W*	*W*	*DW*	*W*
Phosphate	—	*A*	*A*	*A*	*A*	*W*	*W*	*A*	*A*	*A*	*A*	*A*	*A*	*A*
Borate	—	—	*DW*	—	*DW*	—	—	—	*DW*	*DW*	*DW*	*DW*	*DW*	*DW*
Silicate	—	—	—	—	—	—	—	—	—	—	—	*A*	—	*A*
Arsenite	—	—	*A*	*A*	*A*	—	*W*	*A*	*A*	*A*	—	*A*	—	*A*
Chromate	*A*	—	*A*	*A*	*A*	—	*A*	*A*	*A*	*A*	*A*	*DW*	*A*	*A*

W means soluble in water. *A* means soluble in acids. *I*, insoluble in water and acids.
D W, difficultly soluble in water. *D A*, difficultly soluble in acids.

TABLE 8—*Continued.*

SOLUBILITIES OF CHEMICAL COMPOUNDS.

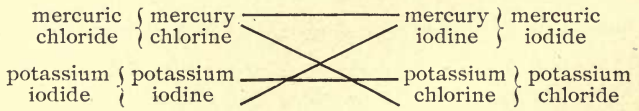
	Aluminum.	Chromic.	Cobaltic.	Nickelic.	Manganous.	Zinc.	Barium.	Strontium.	Calcium.	Magnesium.	Sodium.	Ammonium.	Potassium.	Hydrogen.
Hydrate.....	*A*	*A-I*	*A*	*A*	*A*	*A*	*DW*	*DW*	*DW*	*A*	*W*	*W*	*W*	*W*
Nitrate.....	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*
Carbonate.....	—	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*W*	*W*	*W*	*W*
Acetate.....	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*
Oxalate.....	*A*	*W*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*W*	*W*	*W*	*W*
Cyanide.....	—	*A*	*A*	*A*	*A*	*A*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*
Chloride.....	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*
Bromide.....	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*
Iodide.....	—	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*
Fluoride.....	*A-I*	*W*	*DW*	*DW*	*A*	*DW*	*A*	*A*	*A-I*	*A*	*W*	*W*	*W*	*W*
Sulphide.....	*A*	*A*	*A*	*A*	*A*	*A*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*
Sulphite.....	*A*	*W*	*A*	*A*	*A*	*DW*	*A*	*A*	*W*	*W*	*W*	*W*	*W*	*W*
Sulphate.....	*W*	*W*	*W*	*W*	*W*	*W*	*I*	*I*	*DW*	*W*	*W*	*W*	*W*	*W*
Phosphate.....	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*W*	*W*	*W*	*W*
Borate.....	*DW*	*DW*	*DW*	*W*	*DW*	*DW*	*DW*	*DW*	*DW*	*DW*	*W*	*W*	*W*	*W*
Silicate.....	*A-I*	—	—	—	*A*	*A*	*A*	*A*	*A-I*	*A-I*	*W*	—	*W*	*W-I*
Arsenite.....	—	—	*A*	*A*	*A*	—	*DW*	*DW*	*A*	—	*W*	*W*	*A*	*W*
Chromate.....	—	*A*	*DW*	*W*	*W*	*W*	*A*	*DW*	*DW*	*W*	*W*	*W*	*W*	*W*

W means soluble in water. *A* means soluble in acids. *I*, insoluble in water and acids.
D W, difficultly soluble in water. *DA*, difficultly soluble in acids.

2. *Chemical Displacement.*—The various elements possess different degrees of chemical activity; thus, magnesium enters into combination with other elements with great readiness, while platinum, on the other hand, is almost inactive, forming but few compounds. In consequence of such differences in activity, some elements are able to displace others from compounds already formed. Thus, iron is a more active element than copper, and if a piece of it is placed in a solution of a compound of copper and chlorine, the iron displaces the copper; a compound of iron and chlorine is formed, and copper is liberated in the free state. Again, chlorine is a more active element than oxygen, and if water —a compound of oxygen and hydrogen—is submitted to the action of chlorine, the latter element displaces the oxygen, and a compound of hydrogen and chlorine is the result.

3. *Mutual Exchange.*—When two or more compounds are brought together, there are sometimes conditions that lead to an exchange between them of their various constituents. Thus, if a solution of mercuric chloride (a compound of mercury and chlorine) and potassium iodide (a compound of potassium and iodine) are mixed together, the mercury and iodine, through mutual attraction, unite to form a new compound—mercuric iodide. So, too, the remaining elements, potassium and chlorine, also unite to form a new compound —potassium chloride. Putting it in other words, the mercury has, under these conditions, a greater attraction for iodine than for chlorine, and, consequently, the mercury takes the iodine and hands over the chlorine to the potassium.

The chemical change may be represented thus:



4. *Rearrangement of Atoms.*—This class of chemical action belongs almost exclusively to the domain of organic chemistry. There are many examples in which, without altering the proportions of each element present, fresh compounds may be formed by changing the arrangement of the

atoms; that is, their relative position within the molecule. Such changes may be compared to those in which the same letters by transposition may be caused to form two different words.

5. *Direct Decomposition.*—The separation of complex molecules into simpler ones, to which all analytical reactions belong; as, for instance:

$$\underset{\text{sulphuric acid}}{H_2SO_4} = \underset{\text{water}}{H_2O} + \underset{\text{sulphuric oxide}}{SO_3}$$

STOICHIOMETRY.

78. Stoichiometry considers the numerical relations of atoms, and all calculations that can be based on the atomic weights and volumes are stoichiometrical calculations. They are, consequently, of the greatest importance in analytical and synthetical work.

79. We have seen previously that each atom has its own weight, called the atomic weight, and that these atomic weights are the smallest portions by weight of any simple substance—referred to the atom of hydrogen—that can take part in a chemical change.

A molecule being formed of atoms, a molecular weight is the sum of the weights of the atoms of which it is composed. It is also equal to twice the weight of a given volume of a substance in the state of vapor, compared with the same volume of hydrogen gas.

80. Every chemical reaction is simply an alteration in the association and position of atoms, and all chemical equations that represent such a change represent it as occurring between definite quantities of matter.

As an example, the following equation represents the chemical action occurring when solutions of mercuric chloride and potassium iodide are mixed together.

$$\underset{\text{mercuric chloride}}{HgCl_2} + \underset{\text{potassium iodide}}{2KI} = \underset{\text{mercuric iodide}}{HgI_2} + \underset{\text{potassium chloride}}{2KCl}$$

With the assistance of a table of atomic weights, the chemist learns from this equation that 1 part of mercuric chloride, containing 1 combining proportion of mercury, weighing 200, and 2 combining proportions of chlorine, each weighing 35.5, together with 2 parts of potassium iodide, each containing 1 combining proportion of potassium weighing 39, and 1 of iodine weighing 127, together yield, or produce, 1 part of mercuric iodide, containing 1 combining proportion of mercury, weighing 200, and 2 of iodine, weighing 127 each, and 2 parts of potassium chloride, each consisting of 1 combining proportion of potassium, weighing 39, and 1 of chlorine weighing 35.5. As no chemical change affects the weight of matter, the weight of the quantity of a compound represented by its formula must be the sum of those of the constituent atoms; so, too, the weight of the bodies resulting from a chemical change must be the same as that of the bodies before the change, whatever it may be, had occurred. Also the same number of combining proportions of each element must appear on each side of the equation. Although, given a chemical equation and table of combining weights, it is possible to state what relative weight of each element is concerned in any chemical action, it must never be forgotten that such weights are always determined by chemical analysis, and that from the data thus obtained the equation is compiled. So, too, the series of combining weights is simply a tabular expression of results obtained by practical analytic investigation.

81. Calculation of Percentage Composition.—Knowing the molecular weight of any substance, the number of atoms it contains, and the atomic weight of each constituent atom, we can easily find the percentage composition (that is, the composition of 100 parts of a substance, the form in which the results of analyses are usually expressed).

Let m = molecular weight;
a = atomic weight;
n = number of atoms of a constituent;
x = percentage.

Then, we have the proportion

$$m : a\,n = 100 : x$$

From this proportion we obtain the formula

$$x = \frac{100\,a\,n}{m}. \qquad (1.)$$

Case I.—To find, therefore, the percentage composition of a molecule when the molecular weight, number of atoms, and atomic weights are known, use the following rule:

Rule.—*Multiply the atomic weight by the number of atoms, and the product by 100; divide by the molecular weight, and the quotient will be the percentage amount of that constituent.*

EXAMPLE.—The formula for potassium chlorate is $KClO_3$; assigning to each atom its weight, we have the equation

$$\underset{39}{K} + \underset{35.5}{Cl} + \underset{(16\times 3)}{O_3} = \underset{122.5}{KClO_3}$$

which means that 122.5, being the sum of the constituent atoms, represents the molecular weight of potassium chlorate, and that 122.5 parts of $KClO_3$ contain 48 parts of oxygen, 39 parts of potassium, and 35.5 parts of chlorine. What quantities of these constituents will be found in 100 parts of $KClO_3$?

SOLUTION.—

(1) Oxygen; $x = \dfrac{100\times 16\times 3}{122.5} = 39.18\%$ of oxygen. Ans.

(2) Chlorine; $x = \dfrac{100\times 35.5\times 1}{122.5} = 28.98\%$ of chlorine. Ans.

(3) Potassium; $x = \dfrac{100\times 39\times 1}{122.5} = 31.84\%$ of potassium. Ans.

Total, 100.00%.

82. Other Problems by This Formula.—In formula **1** the four quantities a, n, m, and x are employed. Any three of these being known, the fourth can, of course, be easily found. There are, consequently, three more cases to be considered.

Case II.—To find the number of atoms of the constituents of a compound, that is, the chemical formula of the molecule, having given the percentage amount of the constituents, their

atomic weights and the molecular weight of the compound, formula **1**, by transposition, becomes

$$n = \frac{m\,x}{100\,a}, \qquad \textbf{(2.)}$$

from which we derive the following rule:

Rule.—*Multiply the molecular weight by the percentage of the given constituent, and divide the product by the atomic weight of the constituent, multiplied by 100; the quotient is the number of atoms of that constituent in the molecule.*

EXAMPLE.—Sulphuric acid has a molecular weight of 98; its percentage composition is: hydrogen 2.04 per cent., sulphur 32.65 per cent., and oxygen 65.31 per cent. The atomic weight of hydrogen is 1, that of sulphur 32, and that of oxygen 16. What is the formula of sulphuric acid?

SOLUTION.—The atomic weights of hydrogen, sulphur, and oxygen, are, respectively, 1, 32, and 16; the number of atoms of each element is obtained by formula **2.**

$$n = \frac{98 \times 2.04}{1 \times 100} = 2, \text{ number of atoms of hydrogen;}$$

$$n = \frac{98 \times 32.65}{32 \times 100} = 1, \text{ number of atoms of sulphur;}$$

$$n = \frac{98 \times 65.31}{16 \times 100} = 4, \text{ number of atoms of oxygen.}$$

The molecular formula of sulphuric acid is, therefore, H_2SO_4. Ans.

Case III.—Having the molecular weight, the percentage composition, and the number of atoms of any constituent in the molecule, to find the atomic weight of that constituent, we obtain from formula **1**, by transposition,

$$a = \frac{m\,x}{100\,n}, \qquad \textbf{(3.)}$$

from which we have the following rule:

Rule.—*Multiply the molecular weight by the percentage of the constituent whose weight is desired, and divide the product by the number of atoms multiplied by 100; the quotient is the atomic weight required.*

EXAMPLE.—Mercurous chloride contains 2 atoms of mercury, constituting 84.96 per cent. of the whole; the molecular weight of mercurous chloride being 470.34, what is the atomic weight of mercury?

SOLUTION.—Substituting these values in formula **3**, we have

$a = \frac{470.34 \times 84.96}{2 \times 100} = 199.8$, the atomic weight of mercury. Ans.

Case IV.—Having given the atomic weight of any constituent, its percentage, and the number of atoms of it in the molecule, to find the molecular weight of the compound, we obtain, by a final transposition of formula **1**,

$$m = \frac{100 a n}{x}. \qquad (4.)$$

Whence, the following rule:

Rule.—*Multiply the atomic weight of the given constituent by the number of its atoms, and this product by 100; divide the final product by the percentage of that constituent, and the quotient is the molecular weight.*

EXAMPLE.—One molecule of sulphuric acid contains 1 atom of sulphur, which is 32.65 per cent. of it; the atomic weight of sulphur being 32, what is the molecular weight of sulphuric acid?

SOLUTION.—Substituting these values in formula **4**,

$m = \frac{32 \times 1 \times 100}{32.65} = 98$, the molecular weight of sulphuric acid. Ans.

83. Calculation of an Atomic Group.—It sometimes becomes necessary to calculate the percentage of a group of atoms in a molecule. Formula **1** enables us to do this, using a to indicate the weight of the group, and n the number of such groups in the molecule.

EXAMPLE.—The mineral magnesite $MgCO_3$ is decomposed by heat into MgO and CO_2; how much magnesium oxide MgO is there in 100 parts of magnesite?

SOLUTION.—The molecular weight of MgO is 40, and of $MgCO_3$ is 84. By formula **1**, $\frac{a n \times 100}{m}$, we have

$\frac{40 \times 1 \times 100}{84} = 47.62\%$ of magnesium oxide. Ans.

84. Other Than Percentage Numbers.—In some cases it is necessary to calculate the quantity of a constituent in more or less than 100 parts. The answers to such problems can, of course, be obtained by stating the proportion for

each problem; but they may also be derived from formula **1** by putting y, the quantity of the constituent, in place of x, and z, the quantity of the compound, in place of 100. The formula then becomes

$$y = \frac{a\,n \times z}{m}, \qquad \textbf{(5.)}$$

from which we obtain the following rule:

Rule.—*Multiply the weight of the constituent contained in one molecule by the weight of the compound given in the problem, and divide this product by the molecular weight; the quotient will be the weight of the required constituent.*

EXAMPLE.—How much iodine may be obtained from 311 grams of potassium iodide *KI*, the atomic weight of iodine being 127, and the molecular weight of potassium iodide being 166?

SOLUTION.—Substituting the proper values in formula **5**, we obtain

$$y = \frac{127 \times 311}{166} = 237.9.$$

Hence, 311 grams of potassium iodide yield 237.9 grams of iodine. Ans.

Finally, if we wish to know the quantity of a compound necessary to yield a certain weight of the constituent, we obtain by transposition the following formula:

$$z = \frac{m \times y}{a\,n}. \qquad \textbf{(6.)}$$

EXAMPLE.—How much potassium iodide would be required to yield 59 grams of iodine?

SOLUTION.—Substituting the proper values in formula **6**,

$$z = \frac{166 \times 59}{127} = 77.12 \text{ grams of potassium iodide.} \quad \text{Ans.}$$

85. Calculations From Equations.—The same simple arithmetical principles are applied in calculating the weights of substances that enter into or are produced by chemical changes. The change should be expressed by an equation. The molecular weights of all the participating compounds are calculated from a table of atomic weights and are placed below their formulas. Having got thus far, the problems may be solved by making the following proportion:

The molecular weight of the given substance is to the absolute weight of it given in the problem, as the molecular weight of the required substance is to its required quantity.

Let M = molecular weight of the given substance;
m = molecular weight of the required substance;
W = absolute weight of the given substance;
w = absolute weight of the required substance.

We have the proportion $M : W = m : w$, whence we obtain the following formulas:

$$M = \frac{Wm}{w}. \qquad (7.)$$

$$W = \frac{Mw}{m}. \qquad (8.)$$

$$m = \frac{Mw}{W}. \qquad (9.)$$

$$w = \frac{Wm}{M}. \qquad (10.)$$

Hence, if we know any three of the above values (M, W, m, and w), we can find the fourth by using one of the above formulas. The application of these formulas to the four possible cases is shown by the following examples:

Case I.—Knowing the absolute and molecular weights of the given substance, and also the absolute weight of the required substance, to find the molecular weight of the required substance.

EXAMPLE.—A certain chemical reaction is represented by the following equation:

$Zn(NO_3)_2$	+	K_2CO_3	=	$ZnCO_3$	+	$(KNO_3)_2$
zinc nitrate 189		potassium carbonate 138		zinc carbonate 125		potassium nitrate 202

If 156 grams of zinc nitrate yield 103.17 grams of zinc carbonate, and the molecular weight of zinc carbonate is 125, what is the molecular weight of zinc nitrate?

SOLUTION.—Substituting the values just given in formula **7**, we obtain

$M = \frac{156 \times 125}{103.17} = 189$, the molecular weight of zinc nitrate. Ans.

Case II.—Knowing the molecular weight of both the given and also the required substances, to find the absolute weight of the given substance necessary to yield a certain weight of the required substance.

EXAMPLE.—Sodium nitrate is prepared by the action of hydro-disodium phosphate upon strontium nitrate, according to the following equation:

$$\underset{\substack{\text{strontium}\\\text{nitrate}\\211}}{Sr(NO_3)_2} + \underset{\substack{\text{hydro-di-}\\\text{sodium phosphate}\\142}}{HNa_2PO_4} = \underset{\substack{\text{hydrostrontium}\\\text{phosphate}\\183}}{HSrPO_4} + \underset{\substack{\text{sodium}\\\text{nitrate}\\170}}{(NaNO_3)_2}$$

How much hydro-disodium phosphate will be required in order to yield 309 grams of sodium nitrate?

SOLUTION.—Substituting the proper values in formula **8**, we obtain

$$W = \frac{142 \times 309}{170} = 258.1 \text{ grams of hydro-disodium phosphate. Ans.}$$

Case III.—Knowing the absolute and molecular weights of the given substance and also the absolute weight of the required substance, to find the molecular weight of the latter.

EXAMPLE.—Sodium carbonate is produced by the action of sodium sulphide upon calcium carbonate according to the following equation:

$$\underset{\substack{\text{sodium}\\\text{sulphide}\\78}}{Na_2S} + \underset{\substack{\text{calcium}\\\text{carbonate}\\100}}{CaCO_3} = \underset{\substack{\text{sodium}\\\text{carbonate}\\106}}{Na_2CO_3} + \underset{\substack{\text{calcium}\\\text{sulphide}\\72}}{CaS}$$

103.17 grams of sodium carbonate are obtained from 97.33 grams of calcium carbonate; the molecular weight of calcium carbonate being 100, what is the molecular weight of sodium carbonate?

SOLUTION.—Substituting these values in formula **9**, we obtain

$$m = \frac{100 \times 103.17}{97.33} = 106, \text{ the molecular weight of sodium carbonate. Ans.}$$

Case IV.—Knowing the molecular weights of both the given and the required substance, and also the absolute weight of the given substance, to find the absolute weight of the required substance.

EXAMPLE.—Hydropotassium sulphate is produced by the action of sulphuric acid upon potassium nitrate, according to the following equation:

$$\underset{\substack{\text{potassium}\\\text{nitrate}\\101}}{K(NO_3)} + \underset{\substack{\text{sulphuric}\\\text{acid}\\98}}{H_2SO_4} = \underset{\substack{\text{nitric}\\\text{acid}\\63}}{H(NO_3)} + \underset{\substack{\text{hydropotassium}\\\text{sulphate}\\136}}{HKSO_4}$$

How much hydropotassium sulphate is yielded by the decomposition of 500 grams of potassium nitrate by sulphuric acid ?

SOLUTION.—Substituting the proper values in formula **10**, we obtain $w = \frac{500 \times 136}{101} = 673.27$ grams of hydropotassium sulphate. Ans.

In all the above problems it has been assumed that each molecule of the factor yielded one of the product. If in any reaction this is *not* true, then M and m must represent the sum of the molecular weights expressed in the equation.

86. As all gaseous molecules have the same volume, we are able to read every equation representing a reaction between gaseous bodies, not only quantitatively but also volumetrically.

Thus, the equation

$$\underset{4}{2H_2} + \underset{32}{O_2} = \underset{36}{2H_2O}$$

shows that 2 molecules of hydrogen and 1 molecule of oxygen yield 2 molecules of water. It may also be read: 2 volumes of hydrogen and 1 volume of oxygen yield 2 volumes of water, and calculations may be made from this as well as from the weights.

EXAMPLE.—How much water would be yielded by the explosion of 10 cubic centimeters of hydrogen ?

SOLUTION.—Since 2 volumes of hydrogen yield 2 volumes of water, 10 cubic centimeters of hydrogen will yield 10 cubic centimeters of water in the gaseous state—steam.

87. Relation of the Hydrogen Unit to the Air Unit.—Density has already been defined as the weight of any volume of gas compared with that of the same volume of hydrogen, measured at the same temperature and pressure. (These conditions are always understood in speaking of comparative weights of gases.) The term specific gravity may likewise be defined as the weight of any volume of gas compared with that of the same volume of air. The density of hydrogen is assumed to be 1, and its specific gravity is .0693. It is then obvious that by multiplying the density of any gas by the specific gravity of hydrogen, i. e., .0693, the specific gravity of the gas in question may be obtained, and

that, conversely, by dividing the specific gravity of any gas by .0693 its density may be found.

ILLUSTRATION.—The density of oxygen is 16; its specific gravity, therefore, will be $16 \times .0693 = 1.108$.

The specific gravity of chlorine is 2.46; its density, therefore, is $2.46 \div .0693 = 35.5$.

PRESSURE, VOLUME, AND TEMPERATURE OF GASES.

88. If the temperature of a confined gas remains the same, the pressure and volume will always vary according to *Mariotte's law*, which is as follows:

The temperature remaining the same, the quantity of gas varies inversely as the pressure.

The meaning of this is: If the volume of the gas is diminished to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, etc. of its former volume, the pressure will be increased 2, 3, 5, etc. times, or if the outside pressure is increased 2, 3, 5, etc. times, the volume of the gas will be diminished to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, etc. of its original volume, the temperature remaining constant.

As a necessary consequence of Mariotte's law, it may be stated that *the density of a gas varies directly as the pressure, and inversely as the volume; that is, the density increases as the pressure increases, and decreases as the volume increases.*

Hence, the volume of gases changes with the variation of atmospheric pressure as measured by the barometer; that is, the volume is increased when the barometer falls, and decreased as the barometer rises. The normal pressure to which it is usual to refer gaseous volumes is 760 millimeters of mercury.

If we represent the volume of a gas under a height H of the barometer by V, and under any other height h by v, then, according to the above law, we obtain the proportion:

$$V : v = h : H,$$

which is expressed by the formula:

$$v = \frac{VH}{h}. \qquad (11.)$$

Rule.—*Multiply the given volume of the gas by the given*

height of the barometer; divide the product by 760, and the quotient thus obtained will be the normal volume.

89. Mariotte's law holds when the temperature of the gas remains constant; if the temperature varies, for every increase of temperature there will be a corresponding increase of volume. The law that expresses this change is called *Gay-Lussac's law*, and may be stated as follows:

If the pressure remains constant, every increase of temperature of 1° produces in a given quantity of gas an expansion of $\frac{1}{273}$ of its volume at 0°.

If the pressure remains constant, it will also be found that every decrease of temperature of 1° will cause a decrease of $\frac{1}{273}$ of the volume at 0°.

Let v = original volume of gas;
v_1 = final volume of gas;
t = temperature corresponding to volume v;
t_1 = temperature corresponding to volume v_1.

Then,
$$v_1 = v\left(\frac{273 + t_1}{273 + t}\right). \qquad (12.)$$

Rule.—*The volume of gas after heating (or cooling) equals the original volume multiplied by 273 plus the final temperature, divided by 273 plus the original temperature.*

EXAMPLE 1.—A gas measures 30 cubic centimeters at 0°; what will it measure at 72°?

SOLUTION.—Substituting in formula **12** the proper values, we obtain

$$v_1 = 30\left(\frac{273 + 72}{273 + 0}\right) = 37.912 \text{ c. c.} \quad \text{Ans.}$$

EXAMPLE 2.—A certain quantity of gas measures 36.6 cubic centimeters at 96°; what will it measure at 0°?

SOLUTION.—Substituting in formula **12** the proper values, we obtain

$$v_1 = 36.6\left(\frac{273 + 0}{273 + 96}\right) = 27.08 \text{ c. c.} \quad \text{Ans.}$$

EXAMPLE 3.—A certain quantity of oxygen measures 560 cubic centimeters at 15°; what will the same quantity measure at 95°?

SOLUTION.—Substituting in formula **12** the proper values, we obtain

$$v_1 = 560\left(\frac{273 + 95}{273 + 15}\right) = 715.55 \text{ c. c.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

90. Solve the following:

1. The formula of calcium chloride is $CaCl_2$; its molecular weight is 111. What percentage quantities of the two constituents will be found in 100 parts of calcium chloride? Ans. { 36% of calcium. 64% of chlorine.

2. Potassium chlorate has a molecular weight of 122.5, and is composed of 39.18 per cent. of oxygen, 28.98 per cent. of chlorine, and 31.84 per cent. of potassium. What is the formula of potassium chlorate? Ans. $KClO_3$.

3. Salt $NaCl$ contains 39.32 per cent. of sodium, whose atomic weight is 23. In a molecule of salt there is but 1 atom of sodium. What is the molecular weight of salt? Ans. 58.5.

4. A quantity of ammonia gas had a volume of 350 cubic centimeters when measured at 74°; what volume would it have at 0°?

Ans. 275.3 c. c.

5. The molecular weight of silver nitrate $AgNO_3$ is 170; it contains 63.53 per cent. of silver, and has but 1 atom of silver in a molecule. What is the atomic weight of silver? Ans. 108.

6. Nitric acid is prepared by the action of sulphuric acid on potassium nitrate, according to the equation:

$$\underset{\substack{\text{potassium}\\\text{nitrate}\\101}}{KNO_3} + \underset{\substack{\text{sulphuric}\\\text{acid}\\98}}{H_2SO_4} = \underset{\substack{\text{nitric}\\\text{acid}\\63}}{HNO_3} + \underset{\substack{\text{hydropotassium}\\\text{sulphate}\\136}}{HKSO_4}$$

(*a*) How much potassium nitrate is necessary to yield 36 grams of nitric acid? Ans. 57.71 grams.

(*b*) How much nitric acid may be produced from 500 grams of potassium nitrate? Ans. 311.88 grams.

(*c*) How much sulphuric acid will be required to yield 36 grams of nitric acid? Ans. 56 grams.

7. The density of marsh gas CH_4 is 8; how many grams does a liter of it weigh? Ans. .7168 gram.

8. The specific gravity of hydrogen iodide is 4.41; what is its molecular weight? Ans. 127.26.

CRYSTALLOGRAPHY.

91. Most chemical substances, when they pass from the liquid or gaseous into the solid state, assume some definite geometric form, or are said to *crystallize*. *Crystals* are produced when a substance, such as nitrate of potash, is dissolved in water and the solution is allowed to gradually

evaporate; when a body, such as sulphur, is melted and allowed to solidify by cooling; or when a volatile substance, such as iodine, is vaporized, and the vapor condensed on a cool surface.

One of the most common methods of crystallizing a solid substance consists in dissolving it in hot water and allowing the solution to cool slowly. The more slowly it cools, the larger and more symmetrical are the crystals.

A hot saturated solution is not generally the best for crystallizing, because it deposits the dissolved body too rapidly. Thus, the hot solution of saltpeter prepared as above would solidify to a mass of minute crystals on cooling, but if 100 grams of saltpeter are dissolved in 120 cubic centimeters or 4 measured ounces of boiling water, the solution will form crystals 2 or 3 inches long when slowly cooled in a covered vessel. If the solution is stirred while cooling, the crystals will be very minute, having the appearance of a white powder.

Some solids, however, refuse to crystallize, even from a hot saturated solution, if it is kept absolutely undisturbed. Sodium sulphate affords a good example of this. If to boiling water in a flask the sulphate is added until the water refuses any longer to take it up, it will be found that the water has dissolved more than twice its weight of the salt, and that a solution results which boils at 104.5°. If this solution is allowed to cool in the open flask, an abundant crystallization will take place, for cold water will dissolve only about one-third of its weight of sulphate. But if the flask is tightly corked while the solution is boiling, it may be kept for several days without crystallizing. In this condition we say the solution is *supersaturated.* On withdrawing the cork, the air entering the partly vacuous space above the liquid will be seen to disturb the surface slightly, and from that point beautiful prisms will shoot through the liquid, until the whole has become a nearly solid crystalline mass. A considerable elevation of temperature is observed, consequent on the passage from the liquid to the solid form. If the solution of sodium sulphate is somewhat weaker, containing exactly two-thirds of its weight of the sulphate, it

may be cooled without crystallizing, but a touch with a glass rod will start the crystallization immediately.

92. The crystallization of a supersaturated solution is caused by contact with a crystal of the salt itself. Minute crystals of sodium sulphate are present in the floating dust of the air and cause the crystallization when they fall into the supersaturated solution. A perfectly clean glass rod may be dipped into the solution without causing crystallization, but a rod that has been exposed to air will have some particles of sodium sulphate on it and will start crystallization; if the rod is heated so as to render the sodium sulphate from the dust anhydrous, it will no longer cause crystallization unless it is drawn through the hand.

Air filtered through cotton wool does not cause supersaturated solutions to crystallize. If the solution containing two-thirds of its weight of sodium sulphate is allowed to cool in a flask closed by a cork furnished with two tubes plugged with cotton wool, it will be found that on withdrawing the plugs and blowing through one of the tubes dipping into the solution, crystallization does not take place; but if air is blown by a pair of bellows into the same solution it will crystallize at once.

A striking illustration of the power of unfiltered air to start crystallization is afforded by a solution of alum prepared by saturating a volume of water at 90° and allowing it to cool in a flask, the mouth of which is closed by a plug of cotton wool. In this state it may be kept for weeks without crystallizing, but on withdrawing the plug, crystallization will be seen to commence at once at a few points on the surface immediately under the opening of the neck, and will spread slowly from these, octahedrons of alum of half an inch or more in diameter being built up in a few seconds, the temperature, at the same time, rising very considerably.

In the laboratory, stirring is always resorted to in order to induce crystallization, if it does not take place immediately. Thus, it is usual to test for potassium in a solution by adding tartaric acid, which should cause the formation of minute

crystals of hydropotassium tartrate (cream of tartar), but the test seldom succeeds unless the solutions are stirred briskly with a glass rod.

93. The crystals of sodium sulphate produced in the experiment described above contain, in a state of combination with the salt, more than half their weight of water. Their composition is

anhydrous sodium sulphate Na_2SO_4, 142 parts, or 1 molecule,
water . . . . . H_2O, 180 parts, or 10 molecules,

as expressed by the formula $Na_2SO_4,10H_2O$, or Na_2SO_4, $10aq$ (*aq* being the abbreviation of *aqua*, the Latin name for water). If some of the crystals are pressed between blotting paper to remove adhering water, and left exposed to air, they will gradually *effloresce*, or become covered with a white, opaque powder. This powder is the anhydrous sodium sulphate into which the entire crystals ultimately become converted by exposure to air, since most crystals containing water have their crystalline forms destroyed by the loss of the water, which is commonly spoken of as *water of crystallization*.

Colored salts containing water of crystallization generally change color when the water is removed. The sulphate of copper affords a very good example to this. The beautiful blue prismatic crystals of this salt contain

anhydrous sulphate of copper $CuSO_4$, 159.5 parts, or 1 molecule,
water . . . . . H_2O, 90 parts, or 5 molecules,

which is expressed by the formula $CuSO_4,5H_2O$.

When these are exposed to the air at the ordinary temperature they remain unchanged; but if heated to the boiling point of water they become opaque and are easily crumbled into a grayish-white powder. This powder consists of

anhydrous sulphate of copper $CuSO_4$, 159.5 parts, or 1 molecule,
water . . . . . H_2O, 18 parts, or 1 molecule,

as expressed by the formula $CuSO_4,H_2O$. The four water molecules that have been removed by the heat constituted the water of crystallization, on which the form and color of the sulphate of copper depend. Moistening this white powder with a little water, the blue color is restored with

the evolution of a considerable amount of heat. The one molecule of water that remained after the application of heat cannot be removed unless the salt is heated to 199°, which undoubtedly proves that it is held to the copper sulphate by a more powerful attraction. On this account it is spoken of as *water of constitution*, and in order that the formula of the salt may exhibit the difference between the water of constitution and that of crystallization, it is sometimes written $CuSO_4, H_2O, 4aq$.

We may now define *water of crystallization* as that water which is generally expelled from a crystal at 100°, and which is closely allied with the form and color of the crystal. *Water of constitution* is not generally expelled at 100°, and is in more intimate connection with the chemical properties of the compound forming the crystal.

94. Many native minerals exhibit perfect crystalline forms; though our investigations have not yet disclosed the process of their formation. Judging from their perfect shape we are justified in assuming that the process of their formation must have been a necessarily slow one. Besides their regular shape, these crystalline bodies possess other remarkable properties, that is, a peculiar power of splitting in certain directions more readily than in others, a property that is known as *cleavage;* and in many cases they possess the property of allowing light to pass more readily in certain directions than in others, giving rise to the well known phenomena of *double refraction* (see Art. **158**, *Physics*).

A body assuming two distinct crystalline forms is said to be *dimorphous;* the study of even the more familiar elements includes some very important instances of *dimorphism.* A body that does not occur in crystals is termed *amorphous;* i. e., without crystalline form. Certain complicated structures of the vegetable and animal world exhibit a structure that, although non-crystalline, is not devoid of a certain definite arrangement, and to which the name *organized*, or *cellular*, *structure* has been given.

As a rule, every particular substance possesses a definite

form, in which it will always crystallize, and by which it may be distinguished; when a crystal is formed from aqueous solution, for instance, the most minute particle possesses the same crystalline structure and perfect form as the largest crystals.

Certain substances exhibiting a similarity in their chemical constitution are found to crystallize in the same forms; these are said to be *isomorphous*.

95. Science has succeeded in classifying the thousands of known crystals in *six systems*, to each of which belongs a number of forms having some property in common. In order to classify these different crystals, the existence of certain lines within the crystal, called *axes*, is assumed, around which the form can be symmetrically built up. These axes are assumed to intersect in the center of the crystal, and to pass through from one side to the other.

First, or Regular, System.—Three axes, all equal and at right angles to each other. The simple forms of this system are the *cube*, Fig. 6; the *regular octahedron*, Fig. 7; the *rhombic dodecahedron*, Fig. 8; and the *regular tetrahedron*, Fig. 9.

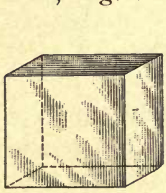


FIG. 6.

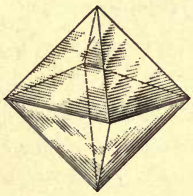


FIG. 7.

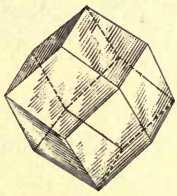


FIG. 8.

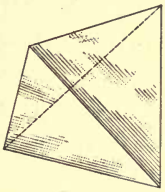


FIG. 9.

The following are examples of substances crystallizing in this system: common salt, diamond, alum, iron pyrites, garnet, fluorspar, etc.

Second, or Quadratic, System.—Three axes, all at right angles, one either shorter or longer than the other two.

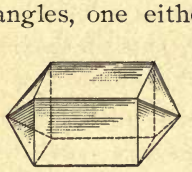


FIG. 10.

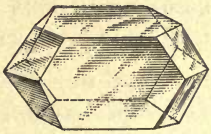


FIG. 11.

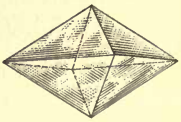


FIG. 12.

The simple forms of this system are the *first right square prism*, Fig. 10; the *second right square prism*, Fig. 11; the *first right square octahedron*, Fig. 12; and the *second right square octahedron*, Fig. 13.

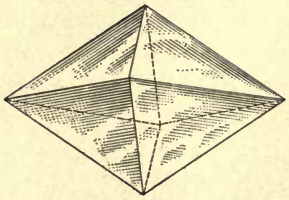


FIG. 13.

In the first square prism, the axes terminate in the center of each of the sides, and in the second square prism, the axes terminate at the intersection of the sides, but this is reversed with regard to the octahedron.

The following are examples of substances crystallizing in this system: stannic oxide, copper pyrites, yellow prussiate of potash, etc.

Third, or Rhombic, System.—Three axes, all unequal, and all at right angles. The chief forms of the crystals in this system are the *right octahedron with rhombic base*, Figs. 14 and 15, and the *right rhombic prism*, Fig. 16.

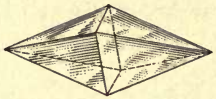


FIG. 14.

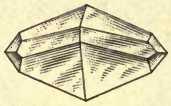


FIG. 15.

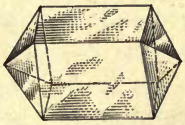


FIG. 16.

Examples of this class are: native sulphur, calcium sulphate, barium sulphate, magnesium sulphate, calcium carbonate, potassium nitrate, etc.

Fourth, or Monoclinic, System.—Three axes, all unequal; two cut each other obliquely, and one is at right angles to the plane of the other two. The *oblique rhombic octahedron*, Fig. 17, belongs to this system.

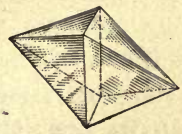


FIG. 17.

Examples of this class are: sodium carbonate, sodium phosphate, borax, cane sugar, ferrous sulphate, etc.

Fifth, or Triclinic, System.—Three axes, all unequal, and all oblique. The *doubly oblique octahedron*, and the *doubly oblique prism*, Fig. 18, are the leading forms of this

system. Copper sulphate $CuSO_4,5H_2O$, boric acid, the mineral albite, potassium bichromate, and a few other sub-

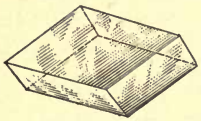


FIG. 18.

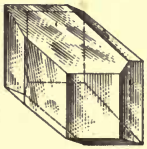


FIG. 19.

stances are found to crystallize in this system, the forms of which are generally very complicated. The crystalline form of copper sulphate is shown in Fig. 19.

Sixth, or Hexagonal, System.—Four axes, three equal and in one plane, making angles of 60°, and one longer or shorter at right angles to the plane of the other three. The

FIG. 20.

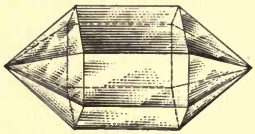


FIG. 21.

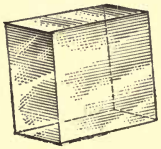


FIG. 22.

hexagonal prism, Fig. 20, the *regular six-sided pyramid*, Fig. 21, and the *rhombohedron*, Fig. 22, are the common forms of the system.

Examples of this class are: graphite, mercuric, sulphide, sodium nitrate, silicic oxide, calcium carbonate, ferrous carbonate, zinc carbonate, etc.

CHEMICAL OPERATIONS.

DEFINITIONS.

96. Allotropy.—When an element occurs in two or more distinct forms it is said to exhibit *allotropy*, and the less common varieties are said to be *allotropic forms* of that element.

97. Chemical Separation.—The chemist frequently finds, in the course of experiments, that it is necessary to

separate bodies from one another ; the processes employed for this purpose are varied, depending on the nature of the substances. As in many future operations it will be necessary to use one or more of them, it is advisable that at this stage they should be studied. As a rule, one of the elements or compounds in a mixture possesses some particular property that the other does not ; thus, one may be soluble in water, and the other insoluble, which at once affords a means of separating them. The principal methods of separation employed are: *solution*, *decantation*, *filtration*, *crystallization*, *evaporation*, *distillation*, *sublimation*, and *ignition*.

98. Solution, Decantation, and Evaporation.—A mixture of two bodies, only one of which is soluble in water, is separated by the action of that *solvent*.

Experiment.—Take some mixture of sand and salt, place in a test tube, add water, and shake up; allow the sand to subside, the clear solution on the top is to be carefully poured off, without disturbing the sediment, into a porcelain basin (evaporating basin). The process is known as *decantation*.

Place a piece of wire gauze on one of the rings of the retort stand, adjusted at a suitable height above the Bunsen burner; put the evaporating basin containing the solution of salt on the gauze and light the burner, keeping the flame small; the water will gradually evaporate and crystals of salt will form. The water has been driven off by *evaporation*. (The object of placing the wire gauze under the basin is to prevent the flame coming in actual contact with and cracking it).

In cases where both bodies are insoluble in water, some substance has to be selected that acts on the one and not the other.

Experiment.—Place some of a mixture of sand and finely powdered marble in a test tube, add some distilled water, and heat. Take out a few drops of the water, and evaporate them on a piece of clean platinum foil; they entirely disappear, or only leave the very slightest stain on the foil. This is the usual test employed to ascertain whether a solvent has dissolved anything or not. Decant off the water and add dilute hydrochloric acid to the mixture; effervescence occurs. When this is over, again take out a few drops of the liquid and evaporate on the foil; a considerable residue will remain. The best way to get a few drops out of a test tube is to put in the end of a glass tube; then close the top with the finger and withdraw it. On removing the finger the small portion it contains runs out.

The separation in this experiment depends on the fact that marble is dissolved by hydrochloric acid, while sand is not affected by it.

99. Filtration.—Cases frequently occur in which decantation is not advisable or practicable for the purpose of separating a liquid and a solid; the solid, for instance, may be so finely divided that it floats in the liquid, giving it a muddy appearance. Under such circumstances the chemist must resort to *filtration.*

EXPERIMENT.—Take a piece of filter paper (soft blotting paper will do, if you have not the especially prepared filter paper at hand), about

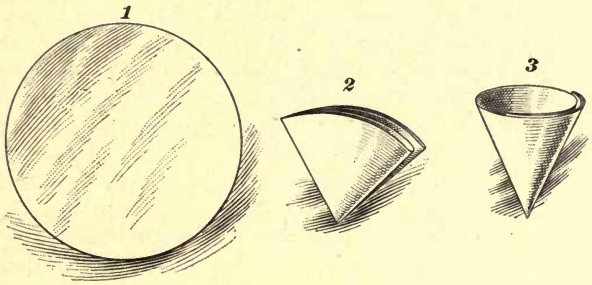


FIG. 23.

$3\frac{1}{2}$ inches in diameter, and double it twice; then open it into a cone, taking three folds of paper on the one side and one on the other, as shown in Fig. 23. Place this cone of paper in a glass funnel, which it will just fit, and moisten it with water. Care must be taken that the point of the folded filter does *not* get broken. Then mix a solution of calcium chloride with some ammonium carbonate; a white precipitate will result. Pour the whole on the filter, holding a glass rod or thin piece of glass tubing against the lip of the beaker or test tube down which the liquid runs (see Fig. 24); the clear liquid, known as the *filtrate*, passes through, and is collected in a bottle or beaker;

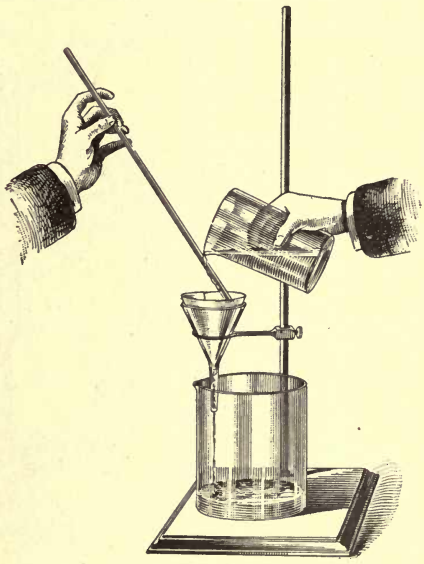


FIG. 24.

the precipitate, which consists of calcium carbonate, remains on the filter. Pour some clean water on it for the purpose of washing. After this has drained off, remove the beaker, punch a hole through the bottom of the filter, wash the precipitate into an evaporating basin and dry it. The filtrate contains ammonium chloride, which may be obtained by evaporation.

100. Crystallization.—Where two bodies are mixed together, both of which are soluble in water, but in different degrees, another plan may be adopted for their separation.

EXPERIMENT.—Take a mixture of about equal parts of potassium chlorate and potassium chloride. Place some in a test tube, and add sufficient water to dissolve the whole on boiling; now allow the solution to cool; tabular crystals consisting of the less soluble potassium chlorate will separate out. When cold, filter these off, wash with cold water, and dry at a gentle heat.

Separation by crystallization is a process largely used for manufacturing purposes and is frequently met in practical work.

101. Distillation.—In the experiment made by evaporation, we have only dealt with solids that remain behind, but in many cases it is the liquid we wish to obtain. In the laboratory, where *pure water* is an essential, it is obtained

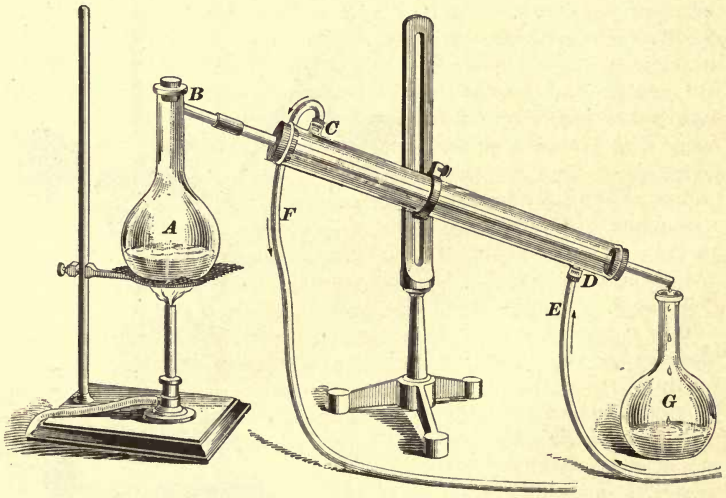


FIG. 25.

by the process of distillation, in which the steam is again condensed and collected.

EXPERIMENT.—Take either an ordinary flask fitted with cork and delivery tube, or, preferably, a flask with side tube, as shown at *A B*, in Fig. 25. Half fill it with brine to which a few drops of coloring matter (ink will do nicely) have been added; fix it in the retort stand and, as shown, connect the tube *B* by a cork to a Liebig's condenser *C D*. The condenser consists of two glass tubes, one fitting inside the other; the steam passes through the inner one, and, as condensed, runs into the flask *G*, placed as a receiver. Through the outer tube a current of cold water is passed, which rapidly condenses the steam. The pipe *E*, conveying the water, is attached to the lower end of the condenser; by pipe *F* the overflow water is led to a drain or into a pan or pail placed conveniently. After some water has collected in the receiver, taste it, and observe that it is perfectly free from salt and coloring matter.

A simple, home-made apparatus is shown in Fig. 26. It consists of two flasks *A* and *C* and a piece of glass tubing about 18 inches long, bent at one end and fitted through a

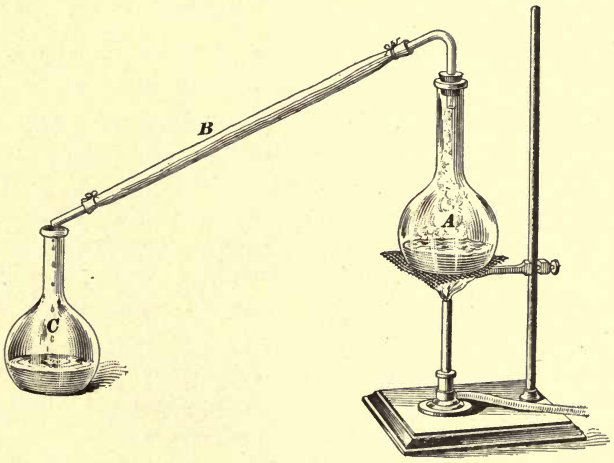


FIG. 26.

cork into flask *A*. To hasten condensation, a piece of wet cloth is tied round the glass tubing *B* and kept wet with cold water as long as the process of distillation lasts.

102. Sublimation.—This is a process somewhat analogous to distillation; it is, in fact, distillation of substances that condense into the solid instead of the liquid state.

EXPERIMENT.—Heat a small portion of a mixture of sand and ammonium chloride in a test tube; dense white fumes are evolved, which condense as a white crust in the upper part of the tube, the sand remaining behind.

Bodies that distil or sublime are called *volatile;* and those that neither distil nor sublime are said to be *fixed.*

103. Ignition.—Occasionally the chemist makes use of the combustibility of bodies, and burns them off from others that are not combustible.

EXPERIMENT.—Mix some sand and lampblack, place some of it on a piece of platinum foil, and keep at bright red heat with the Bunsen burner; the lampblack will soon burn off, leaving the incombustible sand behind.

It may here be mentioned that, when the directions are given to *ignite* a body, it simply means that it is to be heated intensely, but it does *not* always follow that it will *burn.*

104. Combustion.—Every one is familiar with the term *combustion,* which is generally considered as an act of burning. We may now inquire a little more closely into what chemists understand by this expression. It has been previously mentioned that heat is always produced when chemical combination occurs. If, for instance, we add water to sulphuric acid, a considerable elevation of temperature results from the combination. This, of course, is only one of the many cases that could serve as an example, there being many others in which the heat is far more intense.

Whenever the heat caused by chemical union is sufficiently intense to raise the products to a temperature at which they emit light, the act of union is termed *combustion.*

The meaning of the term, however, has been still further widened; thus, certain processes of decay, and also the chemical changes that produce animal heat, are cases of combination with oxygen. These chemical actions are very often termed instances of *slow combustion.*

105. Destructive Distillation.—Destructive distillation may be defined as the resolution of a complex substance into simpler vapors, gases, and liquid products under the influence of heat, out of contact with the air. The operation of destructive distillation will be fully described in the introduction to *Organic Chemistry.*

106. Explosion.—When two or more substances, one of which is inflammable and the others supporters of combustion, are mixed and ignited, combination between these bodies takes place with extreme rapidity. The heat evolved increases the pressure of the gases produced as a result of the chemical action, and these give the surrounding air such a severe shock as to cause a loud detonation.

Extremely rapid combustion, attended with more or less noise, is termed *explosion;* and the mixture that thus explodes is called an *explosive body.*

107. Inflammable Bodies and Supporters of Combustion.—Chemistry has divided all bodies that take part in the act of combustion into *inflammable bodies* and *supporters of combustion.* Thus, when hydrogen burns in oxygen, the former is said to be inflammable, while the latter is said to support the combustion.

These terms, however, are simply relative. We chance to live in an atmosphere of which oxygen is the most active constituent; when a jet of hydrogen is burned, what takes place is that at the opening of the jet the hydrogen and the oxygen of the air come in contact and combination occurs. If the atmosphere, by chance, should consist of hydrogen, then that element would be classified as a supporter of combustion, and oxygen and other similar bodies would be inflammable substances.

108. Nascent Condition of Elements.—The student already knows that the term *atom* is applied to the smallest particles of an element, and that these elements unite to form elemental molecules. Now, we may assume that some portion of the energy that tends to force these atoms to unite

with other atoms is consumed when several atoms of the same kind unite to form an elemental molecule; in fact, it has been found that elements in the atomic state possess a greater combining energy and are more active than the same elements in the form of elemental molecules.

This assumption may be proved to be correct by the following experiment:

EXPERIMENT.—Dissolve some ferric chloride, which possesses a natural red color, and allow hydrogen to bubble up through the solution; you will see that there is no alteration in the appearance of the ferric chloride. Now, add some sulphuric acid and a few pieces of zinc, and you will notice that the color of the ferric chloride very quickly disappears.

Why does this happen? The explanation is very simple: in the second case—that is, when sulphuric acid and zinc were added to the ferric-chloride solution—hydrogen atoms were evolved within the solution, and the liberated atoms, instead of combining to form hydrogen molecules, acted upon the ferric chloride with considerable energy. The following equations show the chemical changes that occur:

(1) $$\underset{\text{zinc}}{Zn} + \underset{\text{sulphuric acid}}{H_2SO_4} = \underset{\text{zinc sulphate}}{ZnSO_4} + \underset{\text{nascent hydrogen}}{2H}$$

(2) $$\underset{\text{nascent hydrogen}}{2H} + \underset{\text{ferric chloride}}{Fe_2Cl_6} = \underset{\text{ferrous chloride}}{2FeCl_2} + \underset{\text{hydrochloric acid}}{2HCl}$$

The reducing power of nascent hydrogen may be more strikingly illustrated by substituting potassium permanganate and hydrochloric acid for ferric chloride and sulphuric acid, in the above experiment. The nascent hydrogen in this case will reduce the potassium permanganate, and thus destroy the strong color it imparts to a solution.

An essential feature of this type of chemical action is that it is caused by an element *at the moment it is liberated from a compound.* Consequently, at the moment of liberation from a compound, an element is said to be *nascent.* (Nascent means "being born.") Most elements possess more active chemical properties when in the nascent state.

When elements are in the molecular state, the force by which the atoms are held together must be overcome before they can enter into fresh combinations. When they are in the nascent condition no such force has to be overcome, and so the atoms are able to enter into combination with much more energy, none of their original energy having previously been absorbed.

109. Oxidation.—The compounds of oxygen with other elements are called oxides, and the act of combination is termed *oxidation.* This process is generally a slow one and its effects are not immediately perceptible. Some familiar examples are the tarnishing or rusting of metals exposed to the air, the gradual decay of wood, the drying of oils in paint, the formation of vinegar from alcohol, the respiration of animals, and combustion.

In all these processes heat is generated; but it is not usually noticed unless it is sufficient to render the matter luminous, which, as we have already seen, is the case only with combustion. The term oxidation is frequently used in another sense. We have seen that many of the elements have variable valence, and certain metals, on account of this, form two series of salts. Thus iron forms ferrous and ferric salts, in which its valence is apparently 2 and 3, respectively. A change from a lower to a higher valence is called oxidation, although oxygen may not be present. Thus we speak of oxidizing ferrous chloride to ferric chloride, or of changing from a lower to a higher state of oxidation.

110. Oxidizing Agents.—An *oxidizing agent* may be defined as a substance that causes an element or compound to combine with oxygen or with other elements of similar character or increases its valence. Oxygen and, to a greater degree, ozone are the most striking examples of oxidizing agents. If, for instance, metallic copper is heated in a current of oxygen, the following reaction takes place:

$$2Cu + O_2 = 2CuO$$

copper oxygen copper oxide

The following substances are some of the most frequently occurring oxidizing agents: *oxygen*, *ozone*, *chlorine*, *bromine*, *potassium chlorate*, *nitric acid*, and *hydrogen peroxide*. (The special properties, etc. of these substances will be fully treated in *Inorganic Chemistry*.)

111. Reaction and Reagent.—The term *reaction* is applied to any chemical change that may be produced, and the substance applied to produce the change is termed a *reagent*.

112. Reducing Agents.—The native compounds or *ores*, from which metals are derived, are in many cases oxides or sulphides of such metals. The process used to obtain these metals in a free or uncombined state is known as the *reduction of the ore* to its metallic state. From this we define a *reducing agent* as a substance that removes oxygen, or elements similar to it, from compounds, or decreases the valence of a substance, changing it from a higher to a lower state of oxidation.

Owing to the activity with which hydrogen combines with oxygen, it is one of the most prominent and powerful reducing agents with which we are acquainted. Thus, if hydrogen is passed over red-hot copper oxide or iron oxide, the metal is obtained in the free state, according to the following equations:

(1) $$\underset{\text{copper oxide}}{CuO} + \underset{\text{hydrogen}}{H_2} = \underset{\text{copper}}{Cu} + \underset{\text{water}}{H_2O}$$

(2) $$\underset{\text{iron oxide}}{Fe_2O_3} + \underset{\text{hydrogen}}{3H_2} = \underset{\text{iron}}{Fe_2} + \underset{\text{water}}{3H_2O}$$

It may here be mentioned, incidentally, that in this case hydrogen, a gas that may be produced by the reducing action of iron on steam, is passed over copper oxide, a product of the oxidizing influence of the oxygen of the air on copper.

The reaction of hydrogen on copper oxide is of great importance, as, in the analysis of many compounds containing

hydrogen, the hydrogen is thus converted into water and weighed as such. In addition to these instances, in which a reducing agent is employed in order to obtain metals in their free state, there are other examples of reduction in which bodies are simply reduced to a lower state of oxidation. Further, the same reducing agents are capable of removing, in whole or in part, as the case may be, chlorine and other similar elements from compounds. Among the most active reducing agents, in addition to hydrogen, are: *carbon*, *carbon monoxide*, *stannous chloride*, and *sulphurous acid.*

ACIDIMETRY AND ALKALIMETRY.

113. Among the various operations performed in analytical work, one of the most important series is that dealing with the estimation of the quantities of acid or alkali contained in various substances. The general practice is to add to a measured quantity of the substance an acid or alkali solution of known strength until the *point of neutrality* is reached. These test solutions are added from graduated vessels known as *burettes*, and the neutral point is determined by the use of some indicating substance. The best known of these are: *litmus*, *methyl-orange*, and *phenol-phthalein.*

114. **Litmus.**—It has already been mentioned that litmus is a vegetable salt, which is turned red by acids and has its original blue color restored by alkalies.

115. **Methyl-Orange.** — Methyl-orange is used in a solution of alcohol, and in alkaline solutions possesses a yellow color, which, however, changes to reddish pink with the slightest excess of acid. Since methyl-orange is not in the least affected by carbonic acid, it is of considerable value in certain estimations.

116. **Phenol-Phthalein.** — Phenol-phthalein is more sensitive than litmus, and in an acid solution does not

produce a color, but with the slightest excess of alkali an intense magenta red is produced.

117. Standard Solutions.—A series of standard solutions is used in acidimetry and alkalimetry, the solutions being of a strength known in the laboratory as *normal solutions.* The *normal acid solutions* are so prepared that at 16° 1 liter of the acid contains 1 gram of replaceable hydrogen.

Accordingly, the *normal sulphuric-acid solution* would contain:

$H_2 + S + O_4$

$(2 + 32 + 64) \div 2 = 49$ grams per liter, or .049 gram per cubic centimeter; and the *normal hydrochloric-acid solution* would contain:

$H + Cl$

$1 + 35.37 = 36.37$ grams per liter, or .036 gram per c. c.

Each liter of the *normal alkali solutions* must contain the chemical equivalent, weighed in grams, of the hydrogen-replacing metal or group of elements.

Therefore, the *normal sodium-hydrate solution* must contain:

$Na + H + O$

$23 + 1 + 16 = 40$ grams per liter, or .040 gram per c. c.

(Sodium and hydrogen are equivalent to each other.)

The *normal barium-hydrate solution* BaH_2O_2 must contain:

$Ba + H_2 + O_2$

$(137 + 2 + 32) \div 2 = 85.5$ grams per liter $= .085$ gram per cubic centimeter.

(The equivalence of barium is twice that of hydrogen.)

118. Equal quantities of any normal acid and alkali solution neutralize each other; and the quantity of any solution of a single acid or alkali that is neutralized by 1 cubic centimeter of a normal solution contains .001 gram of the hydrogen equivalent of its active constituent.

The following description of a typical experiment will fully explain the operations performed:

EXPERIMENT.—The necessary equipment consists of some standard normal solution of sulphuric acid and sodium hydrate, a double burette stand and two burettes, each of 50 cubic centimeters capacity. A burette consists of a glass tube open at the top and drawn to a point at the bottom, over which is drawn a piece of rubber tube with a small glass dropping tube at the end of it. A spring clip (pinch cock) permits the bottom of the burette to be opened or closed at will. The instrument is graduated, having the zero at the top and the scale of figures descending. The burette stand, with the two burettes, is shown in Fig. 27, and the end of a burette in Fig. 28.

After carefully cleaning the two burettes and allowing them to drain, one is rinsed and subsequently filled, above the zero mark, with normal acid solution, and the other one is rinsed and then filled above the zero mark with normal sodium-hydrate solution. Then we press the buttons of the spring clip in each case until the liquid runs down to the zero mark. Next, we pour about 50 cubic centimeters of distilled water into a clean porcelain basin about 4 inches diameter, and run into it from the burette 10 cubic centimeters of normal acid solution. Then we add two or three drops of phenol-phthalein solution and notice that there is no color produced. We read off the height of the soda (sodium-hydrate) solution and make a note of it. Then without consulting the graduations, we run in the soda solution, drop by drop, until one drop imparts a red color to the solution. If we now read off the height of the soda solution in the burette, we should find that exactly 10 cubic centimeters have been used to neutralize the acid.

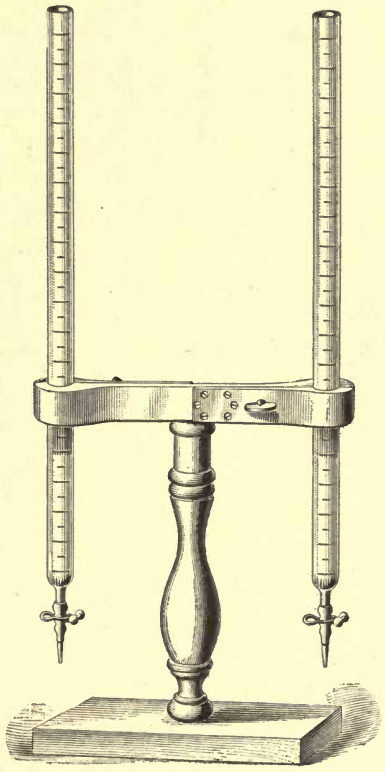


FIG. 27.

Then we repeat the operation, but ask some one to run a quantity of the acid, to us unknown, into the basin. We test with soda solution, reading off and carefully noting the height of the burette before the solution is added, and again as soon as the red color is produced; and then we compare the quantity of the acid and soda solutions, and see whether they agree.

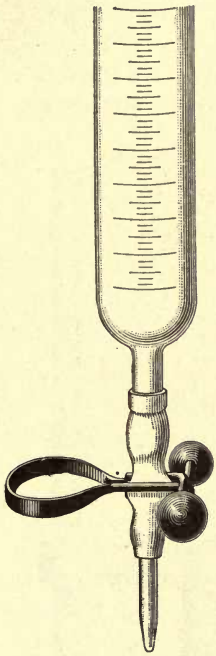


FIG. 28.

We further make a similar determination of alkali by working with an unknown quantity and adding acid until the color of the phenol-phthalein disappears.

Next we determine the percentage of sulphuric acid in a dilute solution of unknown strength. We take a small beaker, carefully wipe it dry, and weigh it. We pour into it about 10 cubic centimeters of the dilute acid, and weigh it again carefully. Then we add some distilled water and two or three drops of phenol-phthalein, and then *titrate* with normal sodium-hydrate solution, and calculate the percentage of acid present. (The verb *titrate* means to measure by means of solutions of known strength, such as are here employed.)

In the next place, we determine the percentage of sodium hydrate in a similar manner in a dilute solution of unknown strength, titrating with normal sulphuric acid.

Next we determine the percentage of sodium carbonate in a sample of common washing soda. We weigh off exactly, say 2 grams of the soda and dissolve it in 50 cubic centimeters of distilled water in a small beaker; we add two or three drops of methyl-orange, and titrate with normal acid until the yellow tint changes to red. With each addition of acid there is an effervescence of carbon dioxide, which, however, does not interfere with the estimation.

These calculations are best illustrated by means of analogous calculations. The following are results actually obtained for this purpose:

DETERMINATION OF SULPHURIC ACID.

Weight of beaker and acid.....	41.92 grams.
Weight of dry beaker	30.23 grams.
Weight of acid.....	11.69 grams.

Final reading of burette	25.9 c. c.
First reading of burette	8.4 c. c.
Normal solution used	17.5 c. c.

Normal acid and soda are equivalent to each other, and therefore 11.69 grams of the acid solution to be tested contains sulphuric acid equal to the quantity in 17.5 cubic centimeters of standard acid; that is,

$$17.5 \times .049 = .8575 \text{ gram.}$$

$.8575 \div 11.69 = .0733$; therefore the solution contains 7.33 per cent. of sulphuric acid.

DETERMINATION OF SODIUM CARBONATE.

Weight of washing-soda..... 2.03 grams.

Final reading of burette	43.3 c. c.
First reading of burette	31.1 c. c.
Acid used for neutralization	12.2 c. c.

It is first necessary to decide what quantity of sodium carbonate is equivalent to the normal solution of sulphuric acid. The following are the formulas and molecular weights of anhydrous and crystallized carbonates, respectively:

Na_2CO_3

$46 + 12 + 48 = 106$, molecular weight of anhydrous sodium carbonates.

$Na_2CO_3 + 10H_2O$

$46 + 12 + 48 + 10(2 + 16) = 286$, molecular weight of crystallized sodium carbonate.

As the molecule contains twice the sodium equivalent, it is evident that half of these quantities—namely, 53 grams per liter of the anhydrous, and 143 grams for the crystallized carbonates—are the strengths for normal solutions equivalent to those of other acids and alkalies. From this it follows that the values for cubic centimeters are, respectively, .053

and .143 gram. The acid taken for neutralization is, therefore, equivalent to:

$$12.2 \times .053 = .447 \text{ gram of } Na_2CO_3.$$
$$12.2 \times .143 = 1.745 \text{ grams of } Na_2CO_3,10H_2O.$$

From these the percentages are readily obtained:

As 2.03 : 100 = .447 : 22.02 per cent. of Na_2CO_3 in sample.
As 2.03 : 100 = 1.745 : 85.96 per cent. of $Na_2CO_3,10H_2O$ in sample.

Further, $100 - 85.96 = 14.04$ per cent. of water and other impurities.

(A purposely contaminated sample had been used.)

LABORATORY DIRECTIONS AND PRACTICAL HINTS.

119. Several of the experiments described in this paper, although illustrating only simple facts, require some skill and experience in chemical manipulations for their successful performance, and the student will probably be content with carefully studying them and grasping the lesson that they convey. It should, however, be mentioned that experimenting is an essential to the study of chemical science, and we advise the student to try to perform as many of the experiments given in the following Instruction Papers as possible.

The student will, after a few unsuccessful trials, soon gain some experience; he should, however, always remember that "*patience* and *carefulness*" are indispensable to the make-up of a good chemist.

If you are not successful with an experiment, try again and again until you succeed.

To facilitate the study of chemistry for those that have not access to a laboratory, The International Correspondence Schools furnish a set of apparatus, and a large number of chemicals and reagents at a very moderate price.

120. List of apparatus for the performance of experiments:

1 Bunsen Burner, with 3-foot Tubing.
1 Nest of 3 Beakers, Nos. 1–3.
3 Flasks, 4, 8, and 16 oz.
2 Wide-Mouth Bottles, ½ gal.
¼ lb. Glass Tubing.
1 Thistle Funnel, 12 in.
1 Glass Funnel, 2 in.
1 Tubulated Retort, 4 oz.
1 Receiver, Plain, 4 oz.
6 Test Tubes, 5 in. × $\frac{5}{8}$ in.
1 Stand.
1 Safety Tube, with Thistle Top.
3 Porcelain Dishes, 3½ in. × 4½ in.
1 Mortar and Pestle, 3¼ in.
1 Square Glass Plate, 4 in.
1 Mohr's Pinch Cock.
1 Pan, Agate Iron, 8 in.
1 Beehive Shelf.
1 Iron Retort Stand, 2 rings.
1 Clamp, Small Universal Movement.
1 Iron Tripod.
1 Balance, sensitive to $\frac{1}{10}$ gr. (1 mg.).
1 Set Weights, 50 g. to 1 mg.
2 Burettes, 50 c. c., with Glass Stop-Cock.
1 Iron Triangle.
1 Pair Forceps, 5 in.
1 Iron Triangle with Clay Tubes.
1 Shallow Iron Sand Bath, 4 in.
1 Brass Blowpipe with Bulb, 8 in.
1 Piece of Charcoal.
1 Piece Platinum Wire.
2 Pieces Wire Gauze, 4 in.
1 doz. Corks, Assorted.
2 feet Rubber Tubing, $\frac{3}{16}$ in.
1 Deflagrating Spoon.
2 Sheets Red and Blue Litmus Paper.
Platinum Foil, 1 in. × 2 in. × $\frac{3}{2000}$ in. thick.
1 Woulff Bottle, 3 necks, 8 oz.
1 French-Clay Crucible, No. 4.
1 Wrought-Nickel Crucible, 20 c. c.
1 Measuring Cylinder, 25 c. c.
1 Graduated Test Tube, 5 c. c.
1 Stirring Rod.
1 Piece Blue Glass.
2 U Tubes, 6 in.
1 Pkg. J. H. Munktell's No. 0 Filters, 9 cm.
1 Pipette, 1 c. c.
1 Pipette, 5 c. c.
1 Thermometer, graduated to 200° C.

121. The room chosen for a laboratory should be of sufficient height, well lighted and ventilated. It should contain a working bench or long table of hard and durable wood, some shelves for bottles and apparatus, gas for heating and lighting, and should, if possible, be close to a sink and water supply.

The student should, in the first place, make himself familiar with the names and uses of his apparatus. The bottles and jars containing chemicals, all properly labeled, should be so placed that they are easily accessible. They should never be removed from or allowed to stand around on the working

bench; the quantity required should be taken in a flask or test tube, and the bottle should be immediately restored to its proper place. A strict adherence to this rule will prevent the loss of time that otherwise would be spent in hunting all over the place for the desired substance, and will save confusion.

In getting the chemicals for an experiment, be sure that the right ones are taken; see that concentrated acids are not employed where dilute acid should be used, and never use C. P. (chemically pure) substances when the commercial ones will answer. Do not allow stoppers or corks to lie about, or get mixed up. Take no more of the substance than you require, but if you happen to have an excess, do not return it to the bottle. It is better to pour it away, as the greatest confusion has been caused by chemicals being replaced in the wrong bottles.

When directions are given that substances should be mixed, solid bodies are first to be powdered in a mortar, and then stirred together until the mixture is as uniform as possible.

Heat is applied by means of the Bunsen burner, and in heating test tubes the heat should be applied gently at first, with a constant rotary motion of the tube, or, if the tube is fixed, of the burner.

If heating a liquid, never let the flame play on the part of the tube above the surface of the liquid. When a solid is heated, if any moisture happens to be present, it condenses in the upper and cooler parts of the tube; when there is the slightest sign of this, hold the tube almost horizontal, but with the hot part rather higher; this prevents the condensed water from running back on the hot glass.

A diary or note book should be conscientiously kept, and every experiment, the changes noticed, and the results should be carefully entered. *This applies with special force to weighing.*

After you have finished your laboratory work, carefully wash all the apparatus used, and wipe with a dry cloth any

water from the retort stand or other apparatus that might rust.

Never use a piece of apparatus until you have carefully dusted it and have convinced yourself that it is perfectly clean.

122. The Bunsen Burner.—The Bunsen burner, shown in Fig. 29, is named after its inventor. It consists of a gas tube *a*, which projects part way into a large tube *b*, called the mixing tube. Air is admitted through holes *c*, which are closed or regulated by means of the collar or slide *d*. The gas issuing from *a* and the streams of air from the holes *c* mingle in the upper part of the tube *b* and form a mixture that will burn over the mouth of the tube. Before lighting the burner, see that the holes *c* at the bottom are open. The flame should be non-luminous; if smoky, it is a sign that the gas is burning at the lower end of the tube of the burner, and it must be extinguished and relighted. This catching fire at the bottom is the result of the entrance of an excess of air; the remedy is to turn on more gas, or partly close the air holes by adjusting the slide so as to restrict the air supply.

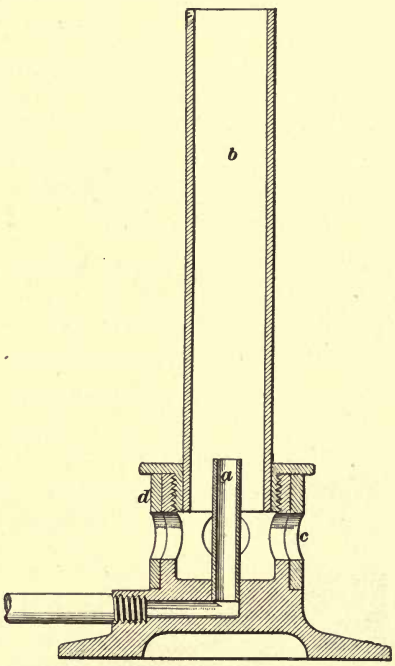


FIG. 29.

123. Glass.—Glass is indispensable to the chemist on account of its transparency, the ease with which it can be

worked in any shape when hot and plastic, and its not being attacked by most chemicals. Its disadvantage is its brittleness. It is particularly liable to break on heating; being a bad conductor of heat, one part of the glass expands on being heated before the neighboring parts become warm.

When boiling water or any other liquid in a glass vessel, always use a tripod and put a piece of wire gauze between the vessel and the flame. The gauze distributes the heat over the bottom of the vessel, and the liquid can be heated until it boils without accident; i. e., without cracking the glass. Or, place some fine sand in an iron tray; put the tray upon the tripod stand and place the glass vessel upon the sand. The heat is distributed by the sand, and the glass vessel can be heated safely. The tray containing the sand is known in the laboratory as a *sand bath.*

124. To Cut Glass Tubes.—It will generally be possible for the student to fit up the apparatus required for his experiments, and he will frequently have to cut a piece of glass tubing to a desired length. This is done very easily. Take a sharp file and make a nick in the glass tube; then

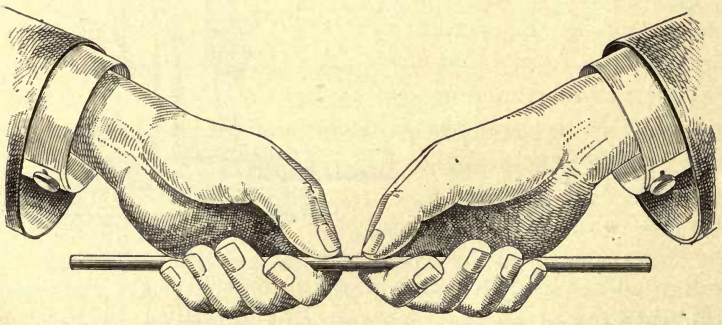


FIG. 30.

hold the tube in both hands; the two forefingers should be together immediately under the nick, the two thumbs above. Turn the wrists outwards and downwards, round the

forefingers as axes, at the same time pulling the hands apart (see Fig. 30).

125. To Bend Glass Tubing.—Hold a piece of glass tubing lengthwise along a flat gas flame (fishtail or batswing, not the Bunsen flame), as shown in Fig. 31. Keep it moving slowly round and round and also to and fro. As soon as it feels slightly soft, hold it by one hand and cease rotating it. The free end will gradually fall (Fig 32). As soon as it is

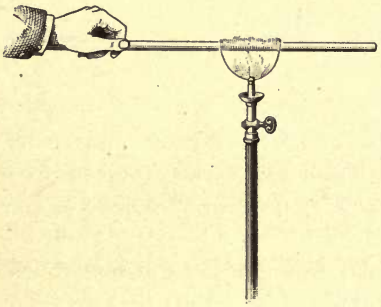


FIG. 31.

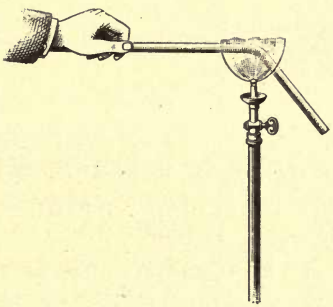


FIG. 32.

bent to the desired angle, remove it from the flame. This method is most frequently used, but as it is likely to cause the glass to bunch at the bend and partially close the tube, it is better to hold the tube by both ends, and bend them upward as soon as the glass softens. A smooth uniform bend is thus obtained.

126. Mistakes to Avoid in Bending Glass.—1. Hold a piece of glass tubing across the Bunsen-burner flame. It becomes hot in two places. When soft bend it. The glass will be found creased at the bend. 2. Hold a piece of glass tubing across an ordinary flat gas flame. The tube is heated this time in one place. Bend it. The tube is flattened on the outside of the bend and thickened on the inside.

127. To Round the Ends of Glass Tubing.—The sharp edges left at the ends of the glass tube must always be carefully rounded, otherwise the tubes will not slide easily

through corks and are apt to cut any rubber tubes that are slipped over them. The ends may be rounded by (*a*) grinding the sharp edge on a file or on a stone, or (*b*) by holding the end in a flame until the glass just melts at its edges.

128. To Draw Out Glass Tubing.—1. Holding a piece of glass tube in both hands, heat it at one place in the Bunsen flame. When soft remove it from the flame and pull. The tube is drawn a little as at *A*, Fig. 33. At *B* it is pulled out

FIG. 33.

FIG. 34.

much finer, because it happens to be hotter, and it has broken there because the pull was too hard. The lump *C* was in the cooler portion of the flame, and never became properly heated.

2. Heat another tube in a flat flame, holding it lengthwise and turning it slowly round and round. Pull the two ends apart; there is no lump, because the tube was heated evenly, but the finely drawn-out tube has softened at once, and bends down, as shown in Fig. 34.

3. Heat another tube as in Experiment 2 until the tube feels limp in the fingers. *Remove it from the flame.* Draw slowly at first, then more quickly, gently rotating each end, until the hands are wide apart. Hold rigidly for a few seconds until the glass stiffens. The glass tube is now drawn out, perhaps a yard long, and has the appearance as shown in Fig. 35. Cut the tube lightly with a sharp file at *A* and *B*, and notice how elastic the section *A B* is; it can be bent nearly to a circle.

FIG. 35.

129. Rubber tubing sometimes becomes stiff when not in use. Warm gently and pull it about before use ; it soon softens. If a rubber tube is too wide for a glass tube it can be made to fit by doubling it back on itself.

130. Pipettes.—Liquids are measured by the flasks and measuring jars that contain them ; it is extremely useful, however, to have vessels that will measure out, or deliver accurately a definite small volume of liquid.

A pipette is a glass tube open at both ends, upon which a cylindrical bulb has been blown (see Fig. 36). There is a circular mark upon the stem. When filled with liquid up to the mark, the pipette will deliver exactly that volume which is marked upon it.

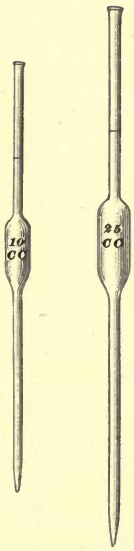


FIG. 36.

How to Use a Pipette.—Place the pointed end of the pipette in water. Suck the water up until it is above the mark. Quickly close the end with the moist forefinger. Pressing gently, rotate the pipette so that the water very slowly flows out until the water level reaches the mark. Then press lightly ; the flow ceases. Transfer the pipette to a beaker, remove the finger, and let the liquid flow out. As soon as the pipette appears to be empty, let it drain for a few seconds with the tip just below the surface of the water in the beaker. The drop of water that is still in the pipette is *left*, and is not blown out.

131. Burettes.—An ordinary burette has been already described in Art. **118**; however, a few practical hints may be useful to the student.

If the figures and graduation on a burette are not easy to read, they may be made more distinct by rubbing them over with a blue pencil.

The surface of a liquid is curved, and the curve is called a *meniscus ;* it is always the lowest point of the meniscus that is measured. Note the following points:

The eye must be on the same level as the curve, otherwise the curve is seen displaced above or below the true position, an effect known as *parallax*. A level position is obtained by

looking straight past the burette at some point known to be on the eye level, or by looking past the curve at its image in a looking glass.

The extreme edge, or lowest position, of the meniscus is, however, sometimes hard to see, but it can easily be made clear. If the background is bright, the meniscus is made dark by holding a dark object just behind and below the surface. The blackened edge of a card slipped over the burette tube answers admirably. At night, however, when the background is dark, a light is put on the working table or a bench below, to illuminate the meniscus.

A SERIES

OF

QUESTIONS AND EXAMPLES

RELATING TO THE SUBJECTS
TREATED OF IN THIS VOLUME.

It will be noticed that the various Question Papers that follow have been given the same section numbers as the Instruction Papers to which they refer. No attempt should be made to answer any of the questions or to solve any of the examples until the Instruction Paper, having the same section number as the Question Paper in which the questions or examples occur, has been carefully studied.

ARITHMETIC.

(ARTS. 1–181. SEC. 1.)

(1) What is arithmetic?

(2) What is a number?

(3) What is the difference between a concrete number and an abstract number?

(4) Define notation and numeration.

(5) Write each of the following numbers in words:
(*a*) 980; (*b*) 605; (*c*) 28,284; (*d*) 9,006,042; (*e*) 850,317,002; (*f*) 700,004.

(6) Represent in figures the following expressions:
(*a*) Seven thousand six hundred. (*b*) Eighty-one thousand four hundred two. (*c*) Five million four thousand seven. (*d*) One hundred eight million ten thousand one. (*e*) Eighteen million six. (*f*) Thirty thousand ten.

(7) What is the sum of $3,290+504+865,403+2,074+81+7$?
Ans. 871,359.

(8) $709+8,304,725+391+100,302+300+909 =$ what?
Ans. 8,407,336.

(9) Find the difference between the following:
(*a*) 50,962 and 3,338; (*b*) 10,001 and 15,339.
Ans. (*a*) 47,624.
(*b*) 5,338.

(10) (*a*) $70,968 - 32,975 = ?$ (*b*) $100,000 - 98,735 = ?$

Ans. { (*a*) 37,993. (*b*) 1,265.

(11) The greater of two numbers is 1,004 and their difference is 49; what is their sum? Ans. 1,959.

(12) From $5,962 + 8,471 + 9,023$ take $3,874 + 2,039$.

Ans. 17,543.

(13) A man willed $125,000 to his wife and two children; to his son he gave $44,675, to his daughter $26,380, and to his wife the remainder. What was his wife's share?

Ans. $53,945.

(14) Find the products of the following:

(*a*) $526,387 \times 7$; (*b*) $700,298 \times 17$; (*c*) $217 \times 103 \times 67$.

Ans. { (*a*) 3,684,709. (*b*) 11,905,066. (*c*) 1,497,517.

(15) If your watch ticks once every second, how many times will it tick in one week? Ans. 604,800 times.

(16) If a monthly publication contains 24 pages in each issue, how many pages will there be in 8 yearly volumes?

Ans. 2,304.

(17) An engine and boiler in a manufactory are worth $3,246. The building is worth three times as much, plus $1,200, and the tools are worth twice as much as the building, plus $1,875. (*a*) What is the value of the building and tools? (*b*) What is the value of the whole plant?

Ans. { (*a*) $34,689. (*b*) $37,935.

(18) Solve the following by cancelation:

(*a*) $\frac{72 \times 48 \times 28 \times 5}{96 \times 15 \times 7 \times 6} = ?$ (*b*) $\frac{80 \times 60 \times 50 \times 16 \times 14}{70 \times 50 \times 24 \times 20} = ?$

Ans. { (*a*) 8. (*b*) 32.

(19) If a mechanic earns $1,500 a year for his labor, and his expenses are $968 per year, in what time can he save enough to buy 28 acres of land at $133 an acre?

Ans. 7 yr.

(20) A freight train ran 365 miles in one week, and 3 times as far, lacking 246 miles, the next week; how far did it run the second week? Ans. 849 mi.

(21) If the driving wheel of a locomotive is 16 feet in circumference, how many revolutions will it make in going from Philadelphia to Pittsburg, the distance between which is 354 miles, there being 5,280 feet in one mile?

Ans. 116,820 rev.

(22) What is the quotient of:

(*a*) 589,824 ÷ 576? (*b*) 369,730,620 ÷ 43,911? (*c*) 2,527,525 ÷ 505? (*d*) 4,961,794,302 ÷ 1,234?

Ans. (*a*) 1,024. (*b*) 8,420. (*c*) 5,005. (*d*) 4,020,903.

(23) A man paid $444 for a horse, wagon, and harness. If the horse cost $264 and the wagon $153, how much did the harness cost? Ans. $27.

(24) What is the product of:

(*a*) 1,024 × 576? (*b*) 5,005 × 505? (*c*) 43,911 × 8,420?

Ans. (*a*) 589,824. (*b*) 2,527,525. (*c*) 369,730,620.

(25) If a man receives 30 cents an hour for his wages, how much will he earn in a year, working 10 hours a day and averaging 25 days per month? Ans. $900.

(26) What is a fraction?

(27) What are the terms of a fraction?

(28) What does the denominator show?

(29) What does the numerator show?

(30) How do you find the value of a fraction?

(31) Is $\frac{13}{8}$ a proper or an improper fraction, and why?

(32) Write three mixed numbers.

(33) Reduce the following fractions to their lowest terms: $\frac{4}{8}$, $\frac{4}{16}$, $\frac{8}{32}$, $\frac{32}{64}$. Ans. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{2}$.

(34) Reduce 6 to an improper fraction whose denominator is 4. Ans. $\frac{24}{4}$.

(35) Reduce $7\frac{7}{8}$, $13\frac{5}{16}$, and $10\frac{3}{4}$ to improper fractions.
Ans. $\frac{63}{8}$, $\frac{213}{16}$, $\frac{43}{4}$.

(36) What is the value of each of the following: $\frac{13}{2}$, $\frac{17}{4}$, $\frac{69}{16}$, $\frac{16}{8}$, $\frac{67}{64}$? Ans. $6\frac{1}{2}$, $4\frac{1}{4}$, $4\frac{5}{16}$, 2, $1\frac{3}{64}$.

(37) Solve the following:

(*a*) $35 \div \frac{5}{16}$; (*b*) $\frac{9}{16} \div 3$; (*c*) $\frac{17}{2} \div 9$; (*d*) $\frac{113}{64} \div \frac{7}{16}$; (*e*) $15\frac{3}{4} \div 4\frac{3}{8}$.

Ans. (*a*) 112. (*b*) $\frac{3}{16}$. (*c*) $\frac{17}{18}$. (*d*) $4\frac{1}{28}$. (*e*) $3\frac{3}{5}$.

(38) $\frac{1}{8} + \frac{2}{8} + \frac{5}{8} = ?$ Ans. 1.

(39) $\frac{1}{4} + \frac{3}{8} + \frac{5}{16} = ?$ Ans. $\frac{15}{16}$.

(40) $42 + 31\frac{5}{8} + 9\frac{7}{16} = ?$ Ans. $83\frac{1}{16}$.

(41) An iron plate is divided into four sections; the first contains $29\frac{3}{4}$ square inches; the second, $50\frac{5}{8}$ square inches; the third, 41 square inches; and the fourth, $69\frac{3}{16}$ square inches. How many square inches are in the plate?
Ans. $190\frac{9}{16}$ sq. in.

(42) Find the value of each of the following:

(*a*) $\dfrac{\frac{7}{3}}{\frac{}{16}}$; (*b*) $\dfrac{\frac{15}{32}}{\frac{5}{8}}$; (*c*) $\dfrac{\frac{4+3}{}}{\frac{2+6}{5}}$.

Ans. (*a*) $37\frac{1}{3}$. (*b*) $\frac{3}{4}$. (*c*) $\frac{7}{40}$.

(43) The numerator of a fraction is 28, and the value of the fraction $\frac{7}{8}$; what is the denominator? Ans. 32.

(44) What is the difference between (*a*) $\frac{7}{8}$ and $\frac{7}{16}$? (*b*) 13 and $7\frac{7}{16}$? (*c*) $312\frac{9}{16}$ and $229\frac{5}{32}$?

Ans. { (*a*) $\frac{7}{16}$. (*b*) $5\frac{9}{16}$. (*c*) $83\frac{13}{32}$.

(45) If a man travels $85\frac{5}{12}$ miles in one day, $78\frac{9}{15}$ miles in another day, and $125\frac{17}{35}$ miles in another day, how far did he travel in the three days?

Ans. $289\frac{211}{420}$ mi.

(46) From $573\frac{4}{5}$ tons take $216\frac{5}{8}$ tons. Ans. $357\frac{7}{40}$ T.

(47) At $\frac{3}{8}$ of a dollar a yard, what will be the cost of $9\frac{1}{4}$ yards of cloth? Ans. $3\frac{15}{32}$ dollars.

(48) Multiply $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{7}{11}$ of $\frac{19}{20}$ of 11 by $\frac{7}{8}$ of $\frac{5}{6}$ of 45.

Ans. $109\frac{13}{128}$.

(49) How many times are $\frac{2}{3}$ contained in $\frac{3}{4}$ of 16?

Ans. 18 times.

(50) Bought $211\frac{1}{4}$ pounds of old lead for $1\frac{7}{8}$ cents per pound. Sold a part of it for $2\frac{1}{2}$ cents per pound, receiving for it the same amount as I paid for the whole. How many pounds did I have left?

Ans. $52\frac{13}{16}$ lb.

(51) Write out in words the following numbers: .08, .131, .0001, .000027, .0108, and 93.0101.

(52) How do you place decimals for addition and subtraction?

(53) Give a rule for multiplication of decimals.

(54) Give a rule for division of decimals.

(55) State the difference between a fraction and a decimal.

(56) State how to reduce a fraction to a decimal.

(57) Reduce the following fractions to equivalent decimals: $\frac{1}{2}$, $\frac{7}{8}$, $\frac{5}{32}$, $\frac{65}{100}$, and $\frac{125}{1000}$.

Ans. .5. .875. .15625. .65. .125.

(58) Solve the following:

(*a*) $\dfrac{32.5+.29+1.5}{4.7+9}$; (*b*) $\dfrac{1.283\times\overline{8+5}}{2.63}$;

(*c*) $\dfrac{\overline{589+27}\times\overline{163-8}}{25+39}$; (*d*) $\dfrac{\overline{40.6+7.1}\times(3.029-1.874)}{6.27+8.53-8.01}$.

Ans. (*a*) 2.5029. (*b*) 6.3418. (*c*) 1,491.875. (*d*) 8.1139.

(59) How many inches in .875 of a foot? Ans. $10\frac{1}{2}$ in.

(60) What decimal part of a foot is $\frac{3}{16}$ of an inch?

Ans. .015625.

(61) A cubic inch of water weighs .03617 of a pound. What is the weight of a body of water whose volume is 1,500 cubic inches? Ans. 54.255 lb.

(62) If by selling a carload of coal for $82.50, at a profit of $1.65 per ton, I make enough to pay for 72.6 feet of fencing at $.50 a foot, how many tons of coal were in the car? Ans. 22 T.

(63) Divide 17,892 by 231, and carry the result to four decimal places. Ans. 77.4545+.

(64) What is the value of the following expression carried to three decimal places:

$$\frac{74.26\times24\times3.1416\times19\times19\times350}{33{,}000\times12\times4}=?$$ Ans. 446.619−.

(65) Express: (*a*) .7928 in 64ths; (*b*) .1416 in 32ds; (*c*) .47915 in 16ths.

Ans. (*a*) $\frac{51}{64}$. (*b*) $\frac{5}{32}$. (*c*) $\frac{8}{16}$.

(66) Work out the following examples:

(*a*) $709.63 - .8514$; (*b*) $81.963 - 1.7$; (*c*) $18 - .18$; (*d*) $1 - .001$; (*e*) $872.1 - (.8721 + .008)$; (*f*) $(5.028 + .0073) - (6.704 - 2.38)$.

Ans. (*a*) 708.7786. (*b*) 80.263. (*c*) 17.82. (*d*) .999. (*e*) 871.2199. (*f*) .7113.

(67) Work out the following:

(*a*) $\frac{7}{8} - .807$; (*b*) $.875 - \frac{3}{8}$; (*c*) $(\frac{5}{32} + .435) - (\frac{21}{100} - .07)$; (*d*) What is the difference between the sum of 33 millionths and 17 thousandths, and the sum of 53 hundredths and 274 thousandths?

Ans. (*a*) .068. (*b*) .5. (*c*) .45125. (*d*) .786967.

(68) What is the sum of .125, .7, .089, .4005, .9, and .000027? Ans. 2.214527.

(69) $927.416 + 8.274 + 372.6 + 62.07938 = ?$

Ans. 1,370.36938.

(70) Add 17 thousandths, 2 tenths, and 47 millionths.

Ans. .217047.

(71) Find the products of the following expressions:

(*a*) $.013 \times .107$; (*b*) $203 \times 2.03 \times .203$; (*c*) $2.7 \times 31.85 \times (3.16 - .316)$; (*d*) $(107.8 + 6.541 - 31.96) \times 1.742$.

Ans. (*a*) .001391. (*b*) 83.65427. (*c*) 244.56978. (*d*) 143.507702.

(72) Solve the following:

(*a*) $(\frac{7}{16} - .13) \times \overline{.625 + \frac{5}{8}}$; (*b*) $(\frac{9}{32} \times .21) - (.02 \times \frac{3}{16})$; (*c*) $(\frac{13}{4} + .013 - 2.17) \times \overline{13\frac{1}{4} - 7\frac{5}{16}}$.

Ans. (*a*) .384375. (*b*) .1209375. (*c*) 6.4896875.

(73) Solve the following:

(*a*) $.875 \div \frac{1}{2}$; (*b*) $\frac{7}{8} \div .5$; (*c*) $\dfrac{.375 \times \frac{1}{4}}{\frac{5}{16} - .125}$.

Ans. (*a*) 1.75. (*b*) 1.75. (*c*) .5.

(74) Find the value of the following expression:

$$\frac{\dfrac{1.25 \times 20 \times 3}{87 + (11 \times 8)}}{459 + 32}.$$

Ans. $210\frac{3}{7}$.

(75) From 1 plus .001 take .01 plus .000001.

Ans. .990999.

ARITHMETIC.

(ARTS. 1–168. SEC. 2.)

(1) What is 25 per cent. of 8,428 lb.? Ans. 2,107 lb.

(2) What is 1 per cent. of $100? Ans. $1.

(3) What is $\frac{1}{2}$ per cent. of $35,000? Ans. $175.

(4) What per cent. of 50 is 2? Ans. 4%.

(5) What per cent. of 10 is 10? Ans. 100%.

(6) Solve the following:

(*a*) Base = $2,522 and percentage = $176.54. What is the rate? (*b*) Percentage = 16.96 and rate = 8 per cent. What is the base? (*c*) Amount = 216.7025 and base = 213.5. What is the rate? (*d*) Difference = 201.825 and base = 207. What is the rate?

Ans. (*a*) 7%. (*b*) 212. (*c*) $1\frac{1}{2}$%. (*d*) $2\frac{1}{2}$%.

(7) A farmer gained 15% on his farm by selling it for $5,500. What did it cost him? Ans. $4,782.61.

(8) A man receives a salary of $950. He pays 24% of it for board, $12\frac{1}{2}$% of it for clothing; and 17% of it for other expenses. How much does he save in a year? Ans. $441.75.

(9) If $37\frac{1}{2}$ per cent. of a number is 961.38, what is the number? Ans. 2,563.68.

(10) A man owns $\frac{3}{4}$ of a property. 30% of his share is worth $1,125. What is the whole property worth?

Ans. $5,000.

(11) What sum diminished by 35% of itself equals $4,810?

Ans. $7,400.

(12) A merchant's sales amounted to $197.55 on Monday, and this sum was 12½% of his sales for the week. How much were his sales for the week? Ans. $1,580.40.

(13) The distance between two stations on a certain railroad is 16.5 miles, which is 12½% of the entire length of the road. What is the length of the road? Ans. 132 mi.

(14) After paying 60% of my debts I find that I still owe $35. What was my whole indebtedness? Ans. $87.50.

(15) Reduce 28 rd. 4 yd. 2 ft. 10 in. to inches.

Ans. 5,722 in.

(16) Reduce 5,722 in. to higher denominations.

Ans. 28 rd. 4 yd. 2 ft. 10 in.

(17) How many seconds in 5 weeks and 3.5 days?

Ans. 3,326,400 sec.

(18) How many pounds, ounces, pennyweights, and grains are contained in 13,750 gr.?

Ans. 2 lb. 4 oz. 12 pwt. 22 gr.

(19) Reduce 4,763,254 links to miles.

Ans. 595 mi. 32 ch. 54 li.

(20) Reduce 764,325 cu.in. to cu.yd.

Ans. 16 cu.yd. 10 cu.ft. 549 cu.in.

(21) What is the sum of 2 rd. 2 yd. 2 ft. 3 in.; 4 yd. 1 ft. 9 in.; 2 ft. 7 in.? Ans. 3 rd. 2 yd. 2 ft. 1 in.

(22) What is the sum of 3 gal. 3 qt. 1 pt. 3 gi.; 6 gal. 1 pt. 2 gi.; 4 gal. 1 gi.; 8 qt. 5 pt.? Ans. 16 gal. 3 qt. 2 gi.

(23) What is the sum of 240 gr. 125 pwt. 50 oz. and 3 lb.?

Ans. 7 lb. 8 oz. 15 pwt.

(24) What is the sum of 11° 16′ 12″; 13° 19′ 30″; 20° 25″; 26′ 29″; 10° 17′ 11″? Ans. 55° 19′ 47″.

(25) What is the sum of 130 rd. 5 yd. 1 ft. 6 in.; 215 rd. 2 ft. 8 in.; 304 rd. 4 yd. 11 in.? Ans. 2 mi. 10 rd. 5 yd. 7 in.

(26) What is the sum of 21 A. 67 sq.ch. 3 sq.rd. 21 sq.li.; 28 A. 78 sq.ch. 2 sq.rd. 23 sq.li.; 47 A. 6 sq.ch. 2 sq.rd. 18 sq.li.; 56 A. 59 sq.ch. 2 sq.rd. 16 sq.li.; 25 A. 38 sq.ch. 3 sq.rd. 23 sq.li.; 46 A. 75 sq.ch. 2 sq.rd. 21 sq.li.?
Ans. 255 A. 3 sq.ch. 14 sq.rd. 122 sq.li.

(27) From 20 rd. 2 yd. 2 ft. 9 in. take 300 ft.
Ans. 2 rd. 1 yd. 2 ft. 9 in.

(28) From a farm containing 114 A. 80 sq.rd. 25 sq.yd., 75 A. 70 sq.rd. 30 sq.yd. are sold. How much remains?
Ans. 39 A. 9 sq.rd. 25¼ sq.yd.

(29) From a hogshead of molasses, 10 gal. 2 qt. 1 pt. are sold at one time, and 26 gal. 3 qt. at another time. How much remains? Ans. 25 gal. 2 qt. 1 pt.

(30) If a person were born June 19, 1850, how old would he be August 3, 1892? Ans. 42 yr. 1 mo. 14 da.

(31) A note was given August 5, 1890, and was paid June 3, 1892. What length of time did it run?
Ans. 1 yr. 9 mo. 28 da.

(32) What length of time elapsed from 16 min. past 10 o'clock A. M., July 4, 1883, to 22 min. before 8 o'clock P. M., Dec. 12, 1888? Ans. 5 yr. 5 mo. 8 da. 9 hr. 22 min.

(33) If 1 iron rail is 17 ft. 3 in. long, how long would 51 rails be, if placed end to end? Ans. 53 rd. 1½ yd. 9 in.

(34) Multiply 3 qt. 1 pt. 3 gi. by 4.7.
Ans. 4 gal. 2 qt. 1.7 gi.

(35) Multiply 3 lb. 10 oz. 13 pwt. 12 gr. by 1.5.
Ans. 5 lb. 10 oz. 6 gr.

(36) How many bushels of apples are contained in 9 bbl., if each barrel contains 2 bu. 3 pk. 6 qt.?

Ans. 26 bu. 1 pk. 6 qt.

(37) Multiply 7 T. 15 cwt. 10.5 lb. by 1.7.

Ans. 13 T. 3 cwt. 67.85 lb.

(38) Divide 358 A. 57 sq.rd. 6 sq.yd. 2 sq.ft. by 7.

Ans. 51 A. 31 sq.rd. 8 sq.ft.

(39) Divide 282 bu. 3 pk. 1 qt. 1 pt. by 12.

Ans. 23 bu. 2 pk. 2 qt. $\frac{1}{4}$ pt.

(40) How many iron rails, each 30 ft. long, are required to lay a railroad track 23 mi. long? Ans. 8,096 rails.

(41) How many boxes, each holding 1 bu. 1 pk. 7 qt., can be filled from 356 bu. 3 pk. 5 qt. of cranberries?

Ans. 243 boxes.

(42) If 16 square miles are equally divided into 62 farms, how much land will each contain?

Ans. 165 A. 25 sq.rd. 24 sq.yd. 3 sq.ft. 80+ sq.in.

(43) What is the square of 108? Ans. 11,664.

(44) What is the cube of 181.25? Ans. 5,954,345.703125.

(45) What is the fourth power of 27.61?

Ans. 581,119.73780641.

(46) Solve the following: (*a*) 106^2; (*b*) $(182\frac{1}{8})^2$; (*c*) $.005^2$; (*d*) $.0063^2$; (*e*) 10.06^2.

Ans. (*a*) 11,236. (*b*) 33,169.515625. (*c*) .000025. (*d*) .00003969. (*e*) 101.2036.

(47) Solve the following: (*a*) 753^3; (*b*) 987.4^3; (*c*) $.005^3$; (*d*) $.4044^3$.

Ans. (*a*) 426,957,777. (*b*) 962,674,279.624. (*c*) .000000125. (*d*) .066135317184.

(48) What is the fifth power of 2? Ans. 32.

(49) What is the fourth power of 3? Ans. 81.

(50) What are the values of: (*a*) 67.85^2? (*b*) $967{,}845^2$? (*c*) $(\frac{3}{8})^2$? (*d*) $(\frac{1}{4})^2$?

Ans. (*a*) 4,603.6225. (*b*) 936,723,944,025. (*c*) $\frac{9}{64}$. (*d*) $\frac{1}{16}$.

(51) What is (*a*) the tenth power of 5? (*b*) the fifth power of 9?

Ans. (*a*) 9,765,625. (*b*) 59,049.

(52) Solve the following: (*a*) 1.2^4; (*b*) 11^6; (*c*) 1^7; (*d*) $.01^4$; (*e*) $.1^8$.

Ans. (*a*) 2.0736. (*b*) 1,771,561. (*c*) 1. (*d*) .00000001. (*e*) .00001.

(53) Find the values of the following: (*a*) $.0133^3$; (*b*) 301.011^3; (*c*) $(\frac{1}{8})^3$; (*d*) $(3\frac{3}{4})^3$.

Ans. (*a*) .000002352637. (*b*) 27,273,890.942264331. (*c*) $\frac{1}{512}$. (*d*) $52\frac{47}{64}$, or 52.734375.

(54) In what respect does evolution differ from involution?

NOTE.—In the answers to the following examples, a minus sign after a number indicates that the last digit is not quite as large as the number printed. Thus, 12.497— indicates that the number really is 12.496+, and that the 6 has been made a 7 because the next succeeding figure was 5 or greater. For example, had it been desired to use but three decimal places in example 46 (*b*), the answer would have been written 33,169.516—.

(55) Find the square root of the following: (*a*) 3,486,-784.401; (*b*) 9,000,099.4009; (*c*) .001225.

Ans. (*a*) 1,867.29+. (*b*) 3,000.017—. (*c*) .035.

(56) Extract the square root of (*a*) 10,795.21; (*b*) 73,008.04; (*c*) 90; (*d*) .09.

Ans. (*a*) 103.9. (*b*) 270.2. (*c*) 9.487—. (*d*) .3.

(57) Extract the cube root of (*a*) .32768; (*b*) 74,088; (*c*) 92,416; (*d*) .373248.

Ans. (*a*) .6894+. (*b*) 42. (*c*) 45.212—. (*d*) .72.

(58) Extract the cube root of 2 to six decimal places.

Ans. 1.259921+.

(59) Extract the cube root of (*a*) 1,758.416743; (*b*) 1,191,016; (*c*) $\frac{4}{32}$; (*d*) $\frac{27}{512}$.

Ans. (*a*) 12.07. (*b*) 106. (*c*) $\frac{1}{2}$. (*d*) $\frac{3}{8}$.

(60) Extract the cube root of 3 to six decimal places.

Ans. 1.442250—.

(61) Solve the following: (*a*) $\sqrt{123.21}$; (*b*) $\sqrt{114.921}$; (*c*) $\sqrt{502,681}$; (*d*) $\sqrt{.00041209}$.

Ans. (*a*) 11.1. (*b*) 10.72+. (*c*) 709. (*d*) .0203.

(62) Solve the following: (*a*) $\sqrt[3]{.0065}$; (*b*) $\sqrt[3]{.021}$; (*c*) $\sqrt[3]{8,036,054,027}$; (*d*) $\sqrt[3]{.000004096}$; (*e*) $\sqrt[3]{17}$.

Ans. (*a*) .18663—. (*b*) .2759—. (*c*) 2,003. (*d*) .016. (*e*) 2.5713—.

(63) Solve the following: (*a*) $\sqrt[4]{6,561}$; (*b*) $\sqrt[6]{117,649}$; (*c*) $\sqrt[6]{.000064}$; (*d*) $\sqrt[3]{\frac{8}{8}}$.

Ans. (*a*) 9. (*b*) 7. (*c*) .2. (*d*) .72112+.

(64) Extract the square root of: (*a*) $\frac{1225}{5476}$; (*b*) .3364; (*c*) .1; (*d*) $25.0\frac{3}{4}$; (*e*) $.000\frac{4}{9}$.

Ans. (*a*) $\frac{35}{74}$. (*b*) .58. (*c*) .31623−. (*d*) 5.00749+. (*e*) .02108+.

(65) (*a*) Extract the fourth root of 2 to four decimal places. (*b*) Extract the sixth root of 6.

Ans. (*a*) 1.1892+. (*b*) 1.34801−.

(66) Extract the square root of (*a*) 3.1416 and (*b*) .7854 to four decimal places.

Ans. (*a*) 1.7725−. (*b*) .8862+.

(67) Extract the cube root of (*a*) 3.1416 and (*b*) .5236 to four decimal places.

Ans. (*a*) 1.4646−. (*b*) .8060−.

Find the value of x in the following:

(68) $11.7 : 13 :: 20 : x$. Ans. 22.22+.

(69) (*a*) $20+7 : 10+8 :: 3 : x$; (*b*) $12^2 : 100^2 :: 4 : x$.

Ans. (*a*) 2. (*b*) 277.7+.

(70) (*a*) $\frac{4}{x} = \frac{7}{21}$; (*b*) $\frac{x}{24} = \frac{8}{16}$; (*c*) $\frac{2}{10} = \frac{x}{100}$; (*d*) $\frac{15}{45} = \frac{60}{x}$; (*e*) $\frac{10}{150} = \frac{x}{600}$.

Ans. (*a*) $x = 12$. (*b*) $x = 12$. (*c*) $x = 20$. (*d*) $x = 180$. (*e*) $x = 40$.

(71) $x : 5 :: 27 : 12.5$. Ans. $10\frac{4}{5}$.

(72) $45 : 60 :: x : 24$. Ans. 18.

(73) $x : 35 :: 4 : 7$. Ans. 20.

(74) $9 : x :: 6 : 24$. Ans. 36.

(75) $\sqrt[3]{1,000} : \sqrt[3]{1,331} = 27 : x$. Ans. 29.7.

(76) $64 : 81 = 21^2 : x^2$. Ans. 23.625.

(77) $7 + 8 : 7 = 30 : x$. Ans. 14.

(78) A man whose steps measure 2 ft. 5 in. takes 2,480 steps in walking a certain distance. How many steps of 2 ft. 7 in. will be required for the same distance?

Ans. 2,320 steps.

(79) If a horse travels 12 mi. in 1 hr. 36 min., how far will he travel at the same rate in 15 hr.? Ans. 112.5 mi.

(80) If a column of mercury 27.63 in. high weighs .76 of a pound, what will be the weight of a column of mercury having the same diameter, 29.4 in. high? Ans. .808+ lb.

(81) If 2 gal. 3 qt. 1 pt. of water will last a man 5 da., how long will 5 gal. 3 qt. last him, if he drinks at the same rate?

Ans. 10 da.

(82) Heat from a burning body varies inversely as the square of the distance from it. If a thermometer held 6 ft. from a stove shows a rise in temperature of 24°, how many degrees rise in temperature would it indicate if held 12 ft. from the stove? Ans. 6°.

(83) If a pile of wood 12 ft. long, 4 ft. wide, and 3 ft. high is worth $12, what is the value of a pile of wood 15 ft. long, 5 ft. wide, and 6 ft. high? Ans. $37.50.

(84) If 100 gal. of water run over a dam in 2 hr., how many gallons will run over the dam in 14 hr. 28 min.?

Ans. $723\frac{1}{3}$ gal.

(85) If a cistern 28 ft. long, 12 ft. wide, 10 ft. deep holds 798 bbl. of water, how many barrels of water will a cistern hold that is 20 ft. long, 17 ft. wide, and 6 ft. deep?

Ans. 484½ bbl.

(86) If a railway train runs 444 mi. in 8 hr. 40 min., in what time can it run 1,060 mi. at the same rate of speed?

Ans. 20 hr. 41.44 min.

(87) If sound travels at the rate of 6,160 ft. in 5½ sec., how far does it travel in 1 min.? Ans. 67,200 ft.

(88) If 5 men by working 8 hours a day can do a certain amount of work, how many men by working 10 hours a day can do the same work? Ans. 4 men.

(89) If a man travels 540 miles in 20 days of 10 hours each, how many hours a day must he travel to cover 630 miles in 25 days? Ans. 9⅓ hr.

(90) Referring to example 4, Art. **168,** *Arithmetic*, § 2, what is the horsepower of an engine whose cylinder is 30 inches in diameter, piston speed, 660 feet per minute, and mean effective pressure, 42 pounds per square inch?

Ans. 594 horsepower.

(91) The weight of a cubic inch of cast iron is .261 pound. Referring to Art. **164,** *Arithmetic*, § 2, what is the weight of a solid cast-iron cylinder whose diameter is 12 inches and length is 60 inches? Ans. 1,771.11 lb.

(92) Referring to Art. **167,** *Arithmetic*, § 2, what is the centrifugal force of a 40-pound body revolving in a circle having a radius of 10 inches, at a speed of 18 feet per second? Ans. 484.7 lb.

ELEMENTARY ALGEBRA
AND
TRIGONOMETRIC FUNCTIONS.

(1) How may the signs of all the terms of the denominator of a fraction be changed from + to − or from − to + without altering the value of the fraction?

(2) Solve the equations:

(*a*) $3x+6-2x=7x$; (*b*) $5x-(3x-7)=4x-(6x-35)$; (*c*) $(x+5)^2-(4-x)^2=21x$.

Ans. (*a*) $x=1$. (*b*) $x=7$. (*c*) $x=3$.

(3) Divide:

$3a^3+2-4a^5+7a+2a^6-5a^4+10a^2$ by a^3-1-a^2-2a.

Ans. $2a^3-2a^2-3a-2$.

(4) Multiply:

(*a*) $2+4a-5a^2-6a^3$ by $7a^3$; (*b*) $4x^2-4y^2+6z^2$ by $3x^2y$; (*c*) $3b+5c-2d$ by $6a$.

Ans. (*a*) $14a^3+28a^4-35a^5-42a^6$. (*b*) $12x^4y-12x^2y^3+18x^2yz^2$. (*c*) $18ab+30ac-12ad$.

(5) A man performed a journey of 48 miles in a certain number of hours; but if he had traveled 4 miles more each hour, he would have performed the journey in 6 hours. How many miles did he travel per hour? Ans. 4 miles.

(6) Translate the following algebraic expressions into ordinary language:

$$\sqrt{\frac{a+b+c}{n}}+\sqrt{a}+\frac{b+c}{n}+\sqrt{a+b}+\frac{c}{n}+(a+b)c+a+bc.$$

(7) Find the products of:

(*a*) $\frac{9m^2n^2}{8p^3q^3}$, $\frac{5p^2q}{2xy}$, and $\frac{24x^2y^2}{90mn}$;

(*b*) $3ax+4$ and $\frac{a^2}{9a^3x^2+24a^2x+16a}$.

Ans. (*a*) $\frac{3mnxy}{4pq^2}$. (*b*) $\frac{a}{3ax+4}$.

(8) Find the sine, cosine, and tangent of 17° 27′ 37″.

(9) A vessel containing some water was filled by pouring in 42 more gallons; there was then seven times as much water in the vessel as at first. How much did the vessel hold?

Ans. 49 gallons.

(10) Reduce $\frac{c(a+b)+cd}{(a+b)c}$ to its simplest form.

(11) Find the sine, cosine, and tangent of 63° 4′ 51.8″.

(12) Find the values of the following:

$$\left(c^{-\frac{2}{3}}\right)^{-\frac{1}{3}};\quad \left(m\sqrt{n^3}\right)^{-\frac{1}{2}};\quad \left(cd^{-2}\right)^{\frac{1}{a}}.$$

(13) Simplify $\frac{3+2x}{2-x}-\frac{2-3x}{2+x}+\frac{16x-x^2}{x^2-4}$. Ans. $\frac{1}{x+2}$.

(14) Multiply $\frac{a^2+c^2+ac}{a^2+b^2-c^2-2ab}$ by $\frac{a^2+c^2-b^2-2ac}{a^3c+a^2c^2+ac^3}$.

SUGGESTION.—Factor the numerators and denominators before multiplying.

Ans. $\frac{a+b-c}{ac(a-b+c)}$.

(15) From a^4-b^4 take $5a^3b-7a^2b^2+5ab^3$, and from the result take $3a^4-4a^3b+6a^2b^2+5ab^3-3b^4$.

Ans. $-2a^4-a^3b+a^2b^2-10ab^3+2b^4$.

(16) Sine = .27038, cosine = .27038, and tangent = 2.27038; find the corresponding angles.

(17) (*a*) Give an illustration, not contained in the section on *Algebra*, that will explain the difference between positive and negative quantities. (*b*) In what respects are addition and subtraction different in algebra from addition and subtraction in arithmetic?

(18) (*a*) What is the value of a^0? (*b*) What does $a^0 \div a^{-1}$ equal?

(19) Change the fraction $-\dfrac{c-(a-b)}{c+(a+b)}$ so that the sign before the dividing line will be +.

(20) (*a*) What is the reciprocal of $\frac{24}{49}$? (*b*) Of what number is 700 the reciprocal?

(21) (*a*) From $3a-2b+3c$ take $2a-7b-c-b$. (*b*) Subtract $x^3+y^3-xy^2$ from $2x^3-3x^2y+2xy^3$. (*c*) From $14a+4b-6c-3d$ take $11a-2b+4c-4d$.

Ans. { (*a*) $a+6b+4c$. (*b*) $x^3-3x^2y+xy^2+2xy^3-y^3$. (*c*) $3a+6b-10c+d$.

(22) Find the numerical values of the following expressions when $a=16$, $b=10$, and $x=5$: (*a*) $(ab^2x+2abx)4a$; (*b*) $2\sqrt{4a}-\dfrac{2bx}{a-b}+\dfrac{b-x}{x}$; (*c*) $(b-\sqrt{a})(x^3-b^2)(a^2-b^2)$.

Ans. { (*a*) 614,400. (*b*) $\frac{1}{3}$. (*c*) 23,400.

(23) Find the sum of the following: (*a*) $4xyz-3xyz-5xyz$, $6xyz-9xyz+3xyz$; (*b*) $3a^2+2ab+4b^2$, $5a^2-8ab+b^2$, $-a^2+5ab-b^2$, $18a^2-20ab-19b^2$, and $14a^2-3ab+20b^2$; (*c*) $4mn+3ab-4c$, $3x-4ab+2mn$, and $3m^2-4p$.

Ans. { (*a*) $-4xyz$. (*b*) $39a^2-24ab+5b^2$. (*c*) $6mn-ab-4c+3x+3m^2-4p$.

(24) (*a*) Explain in your own words the difference between a coefficient and an exponent. (*b*) How are coefficients and exponents treated in multiplication, and how in division? (*c*) What is the law of signs in multiplication?

(25) In a right triangle ABC, the hypotenuse $AB = 17.69$ feet, and the side $AC = 9$ ft. 9 in.; find the other three parts.

Ans. $\begin{cases} 56° 33' 15''. \\ 33° 26' 45''. \\ 14 \text{ ft. } 9 \text{ in.} \end{cases}$

(26) Solve the equations:

(*a*) $\dfrac{9x+20}{36} = \dfrac{4(x-3)}{5x-4} + \dfrac{x}{4}$; (*b*) $ax - \dfrac{3a-bx}{2} = \dfrac{1}{2}$;

(*c*) $am - b - \dfrac{ax}{b} + \dfrac{x}{m} = 0.$

Ans. $\begin{cases} (a) & x = 8. \\ (b) & x = \dfrac{3a+1}{2a+b}. \\ (c) & x = bm. \end{cases}$

(27) State how you would read the following expressions:

(*a*) $a^2x^2 + 2a^3b^5 - (a+b)$; (*b*) $\sqrt[3]{x} + y(a-n^2)^{\frac{3}{4}}$;

(*c*) $(m+n)\,(m-n)^2\left(m - \dfrac{n}{2}\right)$.

(28) (*a*) Write a monomial; a binomial; a polynomial. (*b*) In the expression $a + 2ab - b^2$, why cannot the indicated addition and subtraction be performed? (*c*) What operation is indicated between the quantities in $4ac^2d$?

(29) Resolve into their factors:

(*a*) $45x^7y^{10} - 90x^5y^7 - 360x^4y^8$; (*b*) $a^2b^2 + 2abcd + c^2d^2$;

(*c*) $(a+b)^2 - (c-d)^2$.

Ans. (*c*) $(a+b+c-d)\,(a+b-c+d)$.

(30) Remove the symbols of aggregation from the following:

(*a*) $2a - \{3b + [4c - 4a - (2a+2b)] + [3a - (b+c)]\}$;

(*b*) $7a - \{3a - [(2a-5a) + 4a]\}$;

(*c*) $a - \{2b + [3c - 3a - (a+b)] + [2a - (b+c)]\}$.

Ans. $\begin{cases} (a) & 5a - 3c. \\ (b) & 5a. \\ (c) & 3a - 2c. \end{cases}$

(31) In a right triangle $A\,B\,C$, right-angled at C, side $A\,C = 17.5$, side $B\,C = 21.3$; find the other three parts.

Ans. $\begin{cases} 39^\circ\ 24'\ 23''. \\ 50'\ 35'\ 37''. \\ 27.57, \text{nearly}. \end{cases}$

(32) In the formula, $t = \dfrac{W_1 s_1 t_1 + W_2 s_2 t_2}{W_1 s_1 + W_2 s_2}$, it is required to transform the formula so that t_2 will stand alone in the first member. In other words, solve for t_2.

Ans. $t_2 = \dfrac{(W_1 s_1 + W_2 s_2)t - W_1 s_1 t_1}{W_2 s_2}$.

(33) A can do a piece of work in 5 days, B in 6 days, and C in $7\frac{1}{2}$ days; in what time will they do it, working together? Ans. 2 days.

(34) Simplify: (*a*) $\dfrac{\dfrac{1}{1-x} - \dfrac{1}{1+x}}{\dfrac{1}{1-x} + \dfrac{1}{1+x}}$; (*b*) $\dfrac{\dfrac{a^2}{b^3} + \dfrac{1}{a}}{\dfrac{a}{b^2} - \dfrac{a-b}{ab}}$;

(*c*) $\dfrac{1}{x + \dfrac{1}{1 + \dfrac{x+1}{3-x}}}$.

Ans. $\begin{cases} (a)\ \ x. \\ (b)\ \ \dfrac{a+b.}{b}. \\ (c)\ \ \dfrac{4}{3x+3}. \end{cases}$

(35) (*a*) Arrange $a^3b^2 + 2abc + 3 - 7a^2b^3 + 6a^4b^4$ according to the decreasing powers of a; (*b*) according to the increasing powers of b. (*c*) With $a^2 + 1 + 2a^3 + ax$ arranged according to the *increasing* powers of a, should the 1 be placed first or last, and why?

(36) (*a*) Express with radical signs: $x^{\frac{1}{2}}$; $3x^{\frac{1}{3}}y^{-\frac{2}{3}}$; $3x^{\frac{1}{3}}y^{-\frac{3}{5}}z^{\frac{1}{3}}$. (*b*) Clear $a^{-1}b^{\frac{1}{2}} + \dfrac{c^{-2}}{a+b} + (m-n)^{-1} - \dfrac{a^2b^{-2}c}{c^{-3}}$ of negative exponents. (*c*) Express with fractional exponents: $\sqrt[7]{x^6}$; $\sqrt[3]{x^{-4}}$; $(\sqrt[4]{b^5x^2})^3$.

(37) Divide:

(*a*) $\frac{2ax+x^2}{a^3-x^3}$ by $\frac{x}{a-x}$; (*b*) $\frac{6m^2n^2-3n}{4m^4n^2-4m^2n+1}$ by $\frac{3n}{4m^4n^2-1}$;

(*c*) $9+\frac{5y^2}{x^2-y^2}$ by $3+\frac{5y}{x-y}$.

Ans. (*a*) $\frac{2a+x}{a^2+ax+x^2}$. (*b*) $2m^2n+1$. (*c*) $\frac{3x-2y}{x+y}$.

(38) Multiply:

(*a*) $2x^3+2x^2+2x-2$ by $x-1$; (*b*) $x^2-4ax+c$ by $2x+a$; and (*c*) $-a^3+3a^2b-2b^3$ by $5a^2+9ab$.

Ans. (*a*) $2x^4-4x+2$. (*b*) $2x^3-7ax^2+2cx-4a^2x+ac$. (*c*) $-5a^5+6a^4b+27a^3b^2-10a^2b^3-18ab^4$.

(39) Perform the indicated additions:

(*a*) $\frac{x}{x-y}+\frac{x-y}{y-x}$; (*b*) $\frac{x^2}{x^2-1}+\frac{x}{x+1}-\frac{x}{1-x}$;

(*c*) $\frac{3a-4b}{7}-\frac{2a-b+c}{3}+\frac{13a-4c}{12}$.

Ans. (*a*) $\frac{y}{x-y}$. (*b*) $\frac{3x^2}{x^2-1}$. (*c*) $\frac{71a-20b-56c}{84}$.

(40) Factor the following: (*a*) $9x^4+12x^2y^2+4y^4$; (*b*) $49a^4-154a^2b^2+121b^4$; (*c*) $64x^2y^2+64xy+16$.

(41) Divide:

(*a*) $3x^2+x+9x^3-1$ by $3x-1$; (*b*) $a^3-2ab^2+b^3$ by $a-b$; (*c*) $7x^3+58x-24x^2-21$ by $7x-3$.

Ans. (*a*) $3x^2+2x+1$. (*b*) a^2+ab-b^2. (*c*) x^2-3x+7.

(42) Why are letters used in algebra, and in what ways do they differ from figures?

(43) (*a*) Reduce $1+2x-\frac{4x-4}{5x}$ to a fractional form.

(*b*) Change $\frac{3x^2+2x+1}{x+4}$ to a mixed quantity.

Ans. $\begin{cases} (a) & \frac{10x^2+x+4}{5x}. \\ (b) & 3x-10+\frac{41}{x+4}. \end{cases}$

(44) In a right triangle $A\,B\,C$, angle $A = 65°\ 13'\ 29''$, hypotenuse $A\,B = 5\frac{1}{2}$ yards; find the other three parts. Give sides in feet and inches.

Ans. $\begin{cases} 24°\ 46'\ 31''. \\ 14 \text{ ft. } 11\frac{3}{4} \text{ in.} \\ 6 \text{ ft. } 11 \text{ in., nearly.} \end{cases}$

(45) What is the ratio of (*a*) x^4-1 to x^2+1? (*b*) $x^4-2x^2y^2+y^4$ to $x-y$? Ans. (*b*) $(x^2-y^2)\,(x+y)$.

(46) Solve the following:

(*a*) $\frac{10x+3}{3}-\frac{6x-7}{2} = 10(x-1)$;

(*b*) $(a^2+x)^2 = x^2+4a^2+a^4$; (*c*) $\frac{x-1}{x-2}-\frac{x+1}{x+2} = \frac{3}{x^2-4}$.

Ans. $\begin{cases} (a) & x = 1\frac{1}{2}. \\ (b) & x = 2. \\ (c) & x = 1\frac{1}{2}. \end{cases}$

(47) Add $159°\ 27'\ 34.6''$, $25°\ 16'\ 8.7''$, and $3°\ 48'\ 53''$.

(48) Combine the like terms of the expressions:

(*a*) $x+y+z-(x-y)-(y+z)-(-y)$;

(*b*) $(2x-y+4z)+(-x-y-4z)-(3x-2y-z)$;

(*c*) $a-[2a+(3a-4a)]-5a-\{6a-[(7a+8a)-9a]\}$.

Ans. $\begin{cases} (a) & 2y. \\ (b) & z-2x. \\ (c) & -5a. \end{cases}$

(49) Divide: (*a*) $35m^3y+28m^2y^2-14my^3$ by $-7my$; (*b*) $4a^4-3a^5b-a^6b^2$ by a^4; (*c*) $4x^3-8x^5+12x^7-16x^9$ by $4x^2$.

Ans. $\begin{cases} (a) & -5m^2-4my+2y^2. \\ (b) & 4-3ab-a^2b^2. \\ (c) & x-2x^3+3x^5-4x^7. \end{cases}$

(50) A post has $\frac{1}{5}$ of its length in the earth, $\frac{3}{7}$ in the water, and 13 feet in the air. What is its length?

Ans. 35 feet.

(51) In the composition of a quantity of gunpowder, the niter was 10 lb. more than $\frac{2}{3}$ of the whole, the sulphur was $4\frac{1}{2}$ lb. less than $\frac{1}{6}$ of the whole, and the charcoal was 2 lb. less than $\frac{1}{7}$ of the niter. What was the amount of gunpowder and of each of the ingredients?

Ans. { Gunpowder, 69 lb.
Niter, 56 lb.
Sulphur, 7 lb.
Charcoal, 6 lb.

(52) Find the values of the following:

(*a*) $\sqrt[3]{-125x^3y^6z^9}$; (*b*) $\sqrt[4]{10{,}000a^{16}b^{20}c^8}$; (*c*) $\sqrt[5]{243m^{15}n^{20}}$;

(*d*) $\sqrt[5]{-\frac{x^5y^{10}z^{15}}{a^{20}b^{15}c^{10}d^5}}$.

Ans. { (*a*) $-5xy^2z^3$.
(*b*) $\pm 10a^4b^5c^2$.
(*c*) $3m^3n^4$.
(*d*) $-\frac{xy^2z^3}{a^4b^3c^2d}$.

(53) In a triangle $A\,B\,C$, side $A\,B = 70$ feet, side $B\,C = 42$ feet, and angle $A = 36° 10'$. Find the angles B and C and the side $A\,C$.

Ans. { Angle $B = 64° 14'$.
Angle $C = 79° 36'$.
Side $A\,C = 64.1$ ft.

(54) (*a*) What is the supplement of $72° 11' 36''$? (*b*) What is the complement of $22° 34' 17''$?

Ans. { (*a*) $107° 48' 24''$.
(*b*) $67° 25' 43''$.

(55) In a triangle $A\,B\,C$, angle $A = 57° 34.5'$, angle $C = 44° 22.5'$ and side $A\,B = 344$ feet. Find angle B and the sides $A\,C$ and $B\,C$.

Ans. { Angle $B = 78° 3'$.
Side $A\,C = 481.22$ ft.
Side $B\,C = 415.19$ ft.

PHYSICS.

(1) A cubic foot of a certain mineral weighs 127 pounds; what is its specific gravity? Ans. 2.034.

(2) A conductor conveying a current of electricity is placed in a horizontal plane pointing north and south. If the north pole of a compass needle tends to point towards the east when the compass is placed directly under the conductor, in what direction is the current flowing in the conductor?

(3) If the specific gravity of bismuth is 9.823, what is the weight of a cubic inch of it? Ans. .3548 lb.

(4) (*a*) Give a short description of a saccharimeter. (*b*) What is its principal use?

(5) (*a*) What is science? (*b*) How is natural science distinguished from physical science?

(6) What chemical actions take place in a simple voltaic cell in which the electrolyte is sulphuric acid diluted with water and the electrodes are made of copper and zinc?

(7) (*a*) State what you understand by polarization. (*b*) What is meant by the angle of polarization?

(8) Where will a body weigh more, on the top of a high mountain or at the bottom of a deep valley, the bottom of the valley being as far below sea level as the top of the mountain is above?

(9) The air contained in a closed vessel under a pressure of 12 pounds per square inch is heated from 60° to 300°; what is its final tension? Ans. 17.54 lb.

(10) What are magnets, and from what is the name derived?

(11) (*a*) Define gravity. (*b*) Define cohesion.

(12) A charge of 400 units of electricity is imparted to a sphere 4 inches in diameter; what is the electric density of the charge per square inch over the surface?

Ans. 7.9577 units of electricity per square inch.

(13) (*a*) State the laws of reflection. (*b*) Would these laws hold good if the reflecting surface, instead of being a plane, were curved—as a portion of a sphere?

(14) A certain quantity of air under a pressure of $1\frac{1}{2}$ atmospheres and at a temperature of 75° weighs 7.14 pounds. It expands under the action of heat until 1 cubic foot weighs .08 pound. (*a*) What is the new volume? (*b*) What is the new temperature, the pressure remaining the same? (*c*) What was the original volume?

Ans. (*a*) 89.25 cu. ft.
(*b*) 283.887°.
(*c*) 64.188 cu. ft.

(15) (*a*) What is linear expansion? (*b*) What is surface expansion? (*c*) What is cubical expansion?

(16) If $2\frac{1}{4}$ pounds of lead having a temperature of 40° are placed in 4 pounds of water having a temperature of 65° and contained in a cast-iron vessel whose weight is $1\frac{3}{4}$ pounds and temperature is 62°, what will be the temperature of the mixture? Ans. 64.43°.

(17) Define specific gravity.

(18) What is a Leyden jar?

(19) A certain volume of air under a pressure of $3\frac{1}{2}$ atmospheres weighs 13 pounds. After expanding at a constant temperature, the weight of an equal volume is only 2 pounds. What is the final pressure of the air?

Ans. 7.915 lb. per sq. in.

(20) What is an electrophorus?

(21) (*a*) Of what is a mass constituted? (*b*) How is a molecule constituted?

(22) (*a*) Define absolute zero. (*b*) Define absolute temperature.

(23) A compass *C* is placed between the north and south poles of two magnets, as shown in Fig. 1. Towards which pole will the north pole of the compass needle tend to point, and why?

FIG. 1.

(24) The current in a horizontal conductor is flowing from the north towards the south. In what direction will the north pole of a compass needle point if the compass is placed under the conductor?

(25) What do you understand by a liquid body?

(26) If a body weighs 1 pound at a distance 100 miles from the center of the earth, what will it weigh (*a*) at the surface? (*b*) at 100 miles above the surface? Take the earth's radius as 4,000 miles.

Ans. { (*a*) 40 lb. (*b*) 38.073 lb.

(27) Define (*a*) electrostatic; (*b*) electrodynamic.

(28) How far above the surface of the earth will a 2-pound ball weigh 3 ounces? Ans. 9,064 mi.

(29) Referring to electricity, what is meant by a continuous current?

(30) If 4 cubic feet of copper alloy weigh 2,000 pounds, (*a*) what is the specific gravity? (*b*) What is the weight of a cubic inch?

Ans. { (*a*) Sp. Gr. = 8. (*b*) .2894 lb.

(31) Two gilt balls *A* and *B* are statically charged with $+4$ and -24 units of electricity, respectively; if the balls are allowed to touch momentarily and are then placed at a

distance of 2 centimeters from each other, will they attract or repel each other, and what will be the force in dynes exerted between them?

Ans. Repelled with a force of 25 dynes.

(32) State what you understand by inertia.

(33) Under what conditions does electricity appear?

(34) A piece of wood weighs $11\frac{1}{4}$ ounces in the air. It is attached to a piece of marble weighing 5 pounds in the air and 3 pounds 2 ounces in water. Both together weigh 2 pounds 9 ounces in water. What is the specific gravity (*a*) of the wood? (*b*) of the marble?

Ans. { (*a*) .555. (*b*) 2.667.

(35) What will be the sign of the static charge left on the cover of an electrophorus, if the cake is made of sulphur instead of resin and is rubbed with a piece of silk?

(36) A bottle weighs 2 pounds in air and 10 ounces in water. A pound of sugar is put into the bottle, and the bottle then weighs 16 ounces in water. What is the specific gravity of the sugar? Ans. 1.6.

(37) What will be the sign of the static charge developed (*a*) on a glass rod when rubbed with fur? (*b*) on a piece of hard rubber when rubbed with silk? (*c*) on a piece of flannel when rubbed against a piece of amber?

(38) The weight necessary to sink a Nicholson's hydrometer to a fixed point on the rod is 2 pounds $8\frac{1}{2}$ ounces. The weight necessary to sink the hydrometer to this same point when a piece of slate is in the basket is 1 pound 11 ounces, and when the slate is in the upper pan, 12 ounces. (*a*) What is the specific gravity of the slate? (*b*) What is its volume?

(39) For what purposes is an electroscope used?

(40) What is a vacuum? Illustrate it.

(41) A closed vessel fitted with a piston contains oxygen under a pressure of 3 atmospheres. If the piston is so moved that the volume is $2\frac{1}{2}$ times its former volume, what is the

tension of the gas in pounds per square inch, the temperature remaining the same? Ans. 17.64 lb. per sq. in.

(42) For what is the torsion balance used in electrical measurements?

(43) If you are told that the vacuum gauge of a condenser shows 23 inches vacuum, what do you understand by it? What, in that case, is the pressure in the condenser?

(44) Two gilt balls are statically charged with +20 and −5 units of electricity, respectively; with what force in dynes will they attract each other when placed (*a*) at a distance of 1 centimeter from each other? (*b*) at a distance of 5 centimeters?

Ans. { (*a*) 100 dynes. (*b*) 4 dynes.

(45) What is the weight of 100 cubic feet of hydrogen having a temperature of 80° and a tension of 18 pounds per square inch? Ans. .6231 lb.

(46) Four cubic feet of air are heated under a constant pressure from 40° to 115°. What is the resulting volume? Ans. 4.6 cu. ft.

(47) From what is the word *electricity* derived?

(48) The temperature of the discharged air of an air compressor is 120° and the pressure is 40 pounds per square inch; when it has cooled to the temperature of the surrounding air, which is 55°, what is its pressure? Ans. 35.517 lb. per sq. in.

(49) State the principal sources of heat.

(50) Describe Nichol's prism.

(51) What is the weight of a cubic foot of air at 60° under a pressure of 1 atmosphere? Ans. .0763 lb.

(52) State the law of attraction and repulsion of statically charged bodies.

(53) State, in your own words, what you understand by the "temperature of a body."

(54) The temperature of a wrought-iron bar 4 inches square and 20 inches long is raised 1,200°. What is (*a*) its cubical expansion? (*b*) its linear expansion? (*c*) How much larger will the area of a cross-section be?

Ans. { (*a*) 7.903 cu. in. (*b*) .16464 in. (*c*) .2634 sq. in.

(55) What do you understand (*a*) by a plano-convex lens? (*b*) by a concavo-convex diverging lens? (*c*) How are lenses constructed?

(56) What are the physical properties of electricity?

(57) Two vessels, the volumes of which are each $7\frac{1}{2}$ cubic feet, are filled with air; the temperature is the same in both, but the tension in one is 2 atmospheres and in the other 40 pounds per square inch. If all the air in one vessel is compressed into the other, what is the tension of the mixture after it has cooled down to the original temperature?

Ans. 69.4 lb. per sq. in.

(58) Three gases—oxygen, hydrogen, and nitrogen—are mixed together in an iron retort containing 40 cubic feet. The volume and the tension of the oxygen are 12 cubic feet and 1 atmosphere, respectively; of the hydrogen, 10 cubic feet and 2 atmospheres; of the nitrogen, 8 cubic feet and 3 atmospheres. The temperature of the separate gases and of the mixture remaining the same throughout, what is the tension of the mixture? Ans. 20.58 lb. per sq. in.

(59) How much would a steel wire rope 900 feet long shorten, if cooled from 90° to 28°? Ans. 4.01 in.

(60) Define magnetic density.

(61) (*a*) Name the kind of thermometer in general use. Reduce 44° R. (*b*) to the corresponding centigrade temperature; (*c*) to the corresponding Fahrenheit temperature.

Ans. { (*b*) 55° C. (*c*) 131° F.

(62) Define (*a*) good conductors; (*b*) bad conductors.

(63) State what is meant by terrestrial heat.

(64) Show in a rough sketch (*a*) the refracting edge, (*b*) the refracting angle, and (*c*) the base, of a prism.

(65) A compass *C* is placed alongside of a bar magnet opposite the neutral line, as shown in Fig. 2. Towards which pole of the magnet will the south pole of the compass needle tend to point, and why?

FIG. 2.

(66) A vessel containing 13 cubic feet of air, having a temperature of 73° and a tension of 1 atmosphere, is placed in communication with another vessel containing 18 cubic feet of air, at a temperature of 53°, and a tension of 30 pounds per square inch. What is the new temperature, if the tension of the mixture is 20 pounds per square inch?

Ans. −20.65°.

(67) When you rub a piece of sulphur with a piece of flannel, what charge does the sulphur receive?

(68) Forty cubic feet of air having a temperature of 100° and a tension of 90 pounds per square inch, are mixed with 57 cubic feet of air having a temperature of 130° and a tension of 80 pounds per square inch. The tension of the mixture is 120 pounds per square inch and the temperature is 110°. What is the volume? Ans. 67.248 cu. ft.

(69) Change (*a*) −10° F., (*b*) 25° F., and (*c*) 2,200° F. into the corresponding centigrade readings.

Ans. { (*a*) $-23\frac{1}{3}$° C. (*b*) $-3\frac{8}{9}$° C. (*c*) $1{,}204\frac{4}{9}$° C.

(70) Give the names of the known magnetic substances.

(71) Twelve cubic feet of gas are heated from 65° to 390°; what is the increase in volume, the pressure remaining constant? Ans. 7.4286 cu. ft.

(72) Define (*a*) electrical resistance; (*b*) conductivity.

(73) (*a*) Change 798 B. T. U. to calories. (*b*) Change 40 calories to B. T. U.

Ans. { (*a*) 201.515 calories. (*b*) 158.4 B. T. U.

(74) A piston rod is 4 inches in diameter and a steel piston is bored 3.9985 inches in diameter to receive it. To what temperature must the piston be heated, assuming its original temperature to be 80° and the diameter of its bore after heating to be 4.001 inches, to allow the rod to enter freely?

Ans. 184.4°.

(75) If, in the last example, the whole piston were raised to the same temperature as the bore, and its weight were 360 pounds, how many units of heat would be required, assuming a loss of 12% from various causes? Ans. 4,975.6 B. T. U.

(76) In order to find the temperature of a stove fire, a piece of cast iron weighing 1 pound is placed in it. A copper vessel weighing $1\frac{1}{2}$ pounds is partly filled with $3\frac{1}{4}$ pounds of water, the temperature of both being 85°. After the cast iron piece has been placed in the vessel of water, the temperature of the mixture is found to be 128°. What was the temperature of the fire? Ans. 1,252°.

(77) A solution of cane sugar placed in a tube 12 inches long, and as observed through the saccharimeter, requires an angle of rotation of 29° to reproduce entire darkness. What per cent. of sugar is contained in this solution?

Ans. 12.33%.

(78) What is supposed to be the cause of all electrical phenomena?

(79) What do you understand by "specific heat"?

(80) A vessel has a volume of 6.7 cubic feet. A vacuum gauge attached to it shows $17\frac{1}{2}$ inches. How much air at atmospheric pressure will it be necessary to admit to have the vacuum gauge show (*a*) 5 inches? (*b*) 0 inches?

Ans. { (*a*) $2.79\frac{1}{6}$ cu. ft. (*b*) 3.908 cu. ft.

(81) If 4,516 cubic inches of gas having a temperature of

260° are cooled down to a temperature of 80°, the pressure remaining the same, what is its new volume?

Ans. 1.96 cu. ft.

(82) (*a*) What do you understand by dispersion of light? (*b*) Into what colors is light decomposed? (*c*) What inference has been drawn from the presence of Fraunhofer's lines in the solar spectrum?

(83) What is sensible heat?

(84) How many units of heat would be required to vaporize 12 pounds of solid mercury having a temperature of −50°? (Use the same value for the specific heat of solid mercury as given in the table for liquid mercury.)

Ans. 2,229.595 B. T. U.

(85) What is the absolute temperature corresponding to (*a*) 96° F.? (*b*) 32° C.? (*c*) 180° C.? (*d*) 650° F.? (*e*) −40° C.?

(86) (*a*) Name the rays produced by double refraction. (*b*) Name a mineral which possesses the property of double refraction.

(87) What is meant by the term *quantity of heat?*

(88) State what you know of the theory of light.

(89) State what you know about a magnetic field.

(90) (*a*) What do you understand by a translucent body? (*b*) What is a homogeneous medium?

(91) (*a*) What is meant by polarization in opposite directions? (*b*) Which media possess the property of polarization?

(92) How can you explain the quivering of objects seen over very hot bodies?

(93) (*a*) What do you understand by circular polarization? What does it mean if a substance is said to be levo-rotary?

(94) What is the approximate velocity of light?

(95) What do you understand by the *index of refraction?*

(96) State some experiments for detecting the presence of static charges of electricity.

THEORETICAL CHEMISTRY.

(1) Calcium carbonate $CaCO_3$ may be precipitated by heat, the reaction being represented by the following equation:

$$\underset{\text{calcium bicarbonate}}{CaH_2(CO_3)_2} = \underset{\text{calcium carbonate}}{CaCO_3} + \underset{\text{water}}{H_2O} + \underset{\text{carbon dioxide}}{CO_2}$$

How much calcium carbonate would be precipitated by heating 296.3 grams of calcium bicarbonate ?

Ans. 182.9 grams.

(2) Explain what you understand by chemical combination in *definite* and *multiple* proportion.

(3) Give a short account of how you would proceed to weigh a test tube on a chemical balance.

(4) Define the terms *inflammable body* and *supporter of combustion;* explain why these terms are purely relative.

(5) How many atoms are contained (*a*) in a molecule of mercury ? (*b*) in a molecule of ozone ? (*c*) in a molecule of arsenic ?

(6) A white powder is shaken up with distilled water; how would you ascertain whether any of it is dissolved ?

(7) State Avogadro's law and the deductions therefrom.

(8) Define *meta-acids.*

(9) What is meant by the term *allotropy ?*

(10) The analysis of a compound of chlorine and antimony gives 46.61% of chlorine and 53.39% of antimony; the density of this compound. is 114.25. What is the atomic weight of antimony?

(11) Define matter. Describe its general properties and various physical states.

(12) Define combustion, and state what you understand by that term.

(13) Explain the meaning of the terms *analysis* and *synthesis.*

(14) In which way does methyl-orange differ from phenolphthalein?

(15) Which of the following elements are metalloids and which are metals: *Cl*, *K*, *P*, *S*, *N*, *Na*, *As*, *H*, *Fe*?

(16) (*a*) What do you understand by a compound radical? (*b*) How are compound radicals named?

(17) Give examples of oxidizing and reducing agents.

(18) Define force, and state why chemical action may be properly regarded as a force.

(19) What is molecular volume, and how is it expressed?

(20) What is the percentage composition of chromic acid H_2CrO_4?

Ans. { 1.69% of hydrogen.
44.30% of chromium.
54.01% of oxygen.

(21) A certain volume of oxygen, under a barometric pressure of 781 millimeters, measures 542 cubic centimeters. What is its true volume, measured at 760 millimeters?

Ans. 556.97 c. c.

(22) What is chemistry?

(23) A glassful of dirty water is given to you. Describe the plan you would adopt, first, for rendering the water clear, and, secondly, for ascertaining whether or not the clear water contains any dissolved matter.

(24) State the difference between a mechanical mixture and a chemical combination.

(25) Mention the elements that at ordinary temperatures are, respectively, gases and liquids.

(26) State what you know of chemical and physical changes.

(27) How would you prepare salt from sea-water?

(28) An analysis of a body having a molecular weight of 84 gave the following results:

$$\begin{aligned} C &= 14.29\% \\ O &= 57.14\% \\ H &= 1.19\% \\ Na &= 27.38\% \\ \hline & 100.00\% \end{aligned}$$

What is its chemical formula? Ans. $HNaCO_3$.

(29) State what you know about atomicity.

(30) What relation, if any, exists between the atomic weight and the specific heat of bodies?

(31) (*a*) Define the terms *precipitation*, *precipitate*, and *precipitant*. (*b*) Give Berthollet's law concerning precipitation.

(32) How many volumes of carbon dioxide CO_2 will be formed when a mixture of 4 volumes of carbon monoxide CO and 4 volumes of oxygen O is burned? Show the reaction through an equation; state what volume of oxygen, if any, remains unburned.

(33) Explain the meaning of the terms *sublimation*, *volatile*, *fixed*, and *ignition*.

(34) What are the effects on matter of gravitation, cohesion, and chemical attraction, respectively?

(35) Define the terms *acid*, *base*, and *salt*.

(36) Explain the meaning of the term *nascent*.

(37) Give a short account of the metric system of weights and measures.

(38) By what properties is a metal distinguished from a non-metal?

(39) Define the terms *monobasic* and *dibasic* as applied to acids.

(40) What is the percentage composition of borax $Na_2B_4O_7$?

Ans. { 22.77% of sodium.
21.78% of boron.
55.45% of oxygen.

(41) (*a*) What is expressed by a chemical equation? (*b*) How is such an equation constructed?

(42) What do you understand by a fluid ounce?

(43) State what you know about the nomenclature of acids and salts.

(44) What are stoichiometrical calculations?

(45) Define density.

(46) (*a*) How many atoms may a compound molecule contain? (*b*) How are compound molecules classified?

(47) (*a*) State Gay-Lussac's two laws concerning the proportions in which gaseous volumes combine. (*b*) How may these laws be derived from Avogadro's law?

(48) (*a*) In what way may chemical changes occur? (*b*) Illustrate these changes by using the letters A, B, etc. instead of symbols.

(49) Ammonium iodide may be prepared according to the equation:

$$\underset{\text{ammonia}}{NH_3} + \underset{\substack{\text{hydriodic}\\\text{acid}}}{HI} = \underset{\substack{\text{ammonium}\\\text{iodide}}}{NH_4I}$$

If 56.1 grams of ammonia yield 478.5 grams of ammonium iodide, and the molecular weight of ammonium iodide is 145, what is the molecular weight of ammonia? Ans. 17.

(50) Explain the terms *crystal*, *dimorphous*, *amorphous*, *water of crystallization*, and *isomorphous*.

(51) (*a*) Define molecule. (*b*) State how molecules are classified.

(52) What do you understand (*a*) by a symbol? (*b*) by a formula?

(53) (*a*) How do atoms of different valence combine? (*b*) Suppose a pentad and a dyad combine, how many atoms of each are necessary to form a saturated compound?

(54) In what respects does a mixture of copper and sulphur differ after being heated from its condition previous to the application of heat?

(55) A compound, hydrodisodium phosphate, is found on analysis to contain

hydrogen	=	0.7%
sodium	=	32.4%
phosphorus	=	21.8%
oxygen	=	45.1%
		100.0%

Its molecular weight is 142. What is its formula?

Ans. HNa_2PO_4.

(56) What is the simplest way to distinguish between elemental and compound bodies?

(57) Why does chemical combination always occur in either definite or multiple proportion?

(58) Alumina is composed as follows:

aluminum	=	53.4%
oxygen	=	46.6%
		100.0%

Its molecular weight is 103. What is its formula?

Ans. Al_2O_3.

(59) Potassium sulphate may be obtained according to the following equation :

$$\underset{\text{potassium nitrite}}{2\,KNO_2} + \underset{\text{sulphuric acid}}{H_2SO_4} = \underset{\text{potassium sulphate}}{K_2SO_4} + \underset{\text{nitrous acid}}{2\,HNO_2}$$

How much potassium nitrite will be required to yield 392 grams of potassium sulphate? Ans. 383 grams.

(60) Define (*a*) a monad; (*b*) a heptad; (*c*) a pentad.

(61) Give the names of SrO, $NaCl$, P_2O_5, B_2O_3, $SnCl_4$.

(62) A molecule of potassium nitrate contains 1 atom of potassium, which is by weight 38.7% of it. The molecular weight of potassium nitrate is 101. What is the atomic weight of potassium? Ans. 39.

(63) Describe the process of gaseous diffusion, and state the law governing its rate.

(64) What do you understand by an electronegative element?

(65) Define the meaning of the term *perissad.*

(66) State how acids are classified.

(67) A molecule of zinc carbonate contains 1 atom of zinc, which is by weight 52% of it; the molecular weight of zinc carbonate is 125. What is the atomic weight of zinc? Ans. 65.

(68) What will a gas measure at 0°, which at 100° measured 40.1 cubic centimeters? Ans. 29.349 c. c.

(69) State how you would symbolically express the valence of an atom.

(70) (*a*) State the principles that underlie the nomenclature of binary molecules. (*b*) Which termination is always characteristic of a binary molecule?

(71) Why may mass reactions be represented by molecular formulas?

(72) Zinc sulphide ZnS contains 67% of zinc, or 1 atom of zinc in each molecule. The atomic weight of zinc is 65; what is the molecular weight of zinc sulphide? Ans. 97.

(73) Phosphonium iodide PH_4I may be obtained according to the following equation:

$$\underset{\text{hydrogen phosphide}}{PH_3} + \underset{\text{hydriodic acid}}{HI} = \underset{\text{phosphonium iodide}}{PH_4I}$$

If 883.2 grams of hydriodic acid yield 1,117.8 grams of phosphonium iodide, whose molecular weight is 162, what is the molecular weight of hydriodic acid? Ans. 128.

(74) What volume would 350 cubic centimeters of ammonia gas, measured at 74°, have at 94°? Ans. 370.17 c. c.

(75) Define atomic weight.

(76) Write in words the meaning of each of the following symbols: H, O, O_2, $3H_2$.

(77) How many liters of hydrogen weigh 1 gram?

(78) Ammonium nitrate NH_4NO_3 decomposes under the influence of heat into one molecule of hyponitrous oxide N_2O and 2 molecules of water H_2O. How many parts of hyponitrous oxide N_2O are there in 100 parts of ammonium nitrate? Ans. 55 parts.

(79) Antimonous sulphide may be precipitated according to the following equation:

$$\underset{\text{antimonous chloride}}{2SbCl_3} + \underset{\text{hydrogen sulphide}}{3SH_2} = \underset{\text{hydrochloric acid}}{6HCl} + \underset{\text{antimonous sulphide}}{Sb_2S_3}$$

78.57 grams of antimonous sulphide are obtained from 105.6 grams of antimonous chloride, the molecular weight of which is 228.5. What is the molecular weight of antimonous sulphide? Ans. 340.

(80) You are required to neutralize with caustic soda a quantity of sulphuric acid diluted with water. Describe how you would do this, and how you would ascertain that the neutralization has been effected.

(81) State what the *valence* of an atom means.

(82) What data are necessary to determine the atomic weight of an element?

(83) Salt $NaCl$ contains 39.32% of sodium, the atomic weight of which is 23; what is the molecular weight of salt?

Ans. 58.5.

(84) Sodium sulphate may be formed according to the following equation:

H_2SO_4	+	Na_2CO_3	=	Na_2SO_4	+	CO_2	+	H_2O
sulphuric acid		sodium carbonate		sodium sulphate		carbon dioxide		water
98		106		142		44		18

How much sodium sulphate may be yielded by the decomposition of 396.11 grams of sodium carbonate?

Ans. 530.64 grams.

(85) (*a*) What do you understand by the density of a gas? (*b*) The volume of a given weight of a gas has to be calculated at the standard of a temperature and pressure; what does this mean?

(86) What is (*a*) a chemical reaction? (*b*) a chemical reagent?

(87) How much calcium phosphate is required to yield 501 kilograms of phosphorus? The atomic weight of phosphorus is 31, and the molecular weight of calcium phosphate is 310. Ans. 5,010 Kg.

(88) What volume would 420 cubic centimeters of hydrogen, measured at 739 millimeters pressure, occupy at 760 millimeters pressure? Ans. 408.4 c. c.

(89) (*a*) What are the properties common to acids? (*b*) In what respect does sulphuric acid differ from sulphurous acid?

(90) (*a*) How are chemical reactions classified? (*b*) Illustrate by using the letters A, B, etc. instead of symbols.

(91) How much does 1 liter of hydrogen weigh?

(92) What do you understand by molecular stability?

(93) Give the formulas of potassium iodide, lead sulphide, calcium chloride, and cupric oxide.

(94) Read the following equation by weight:

$$\underset{\text{potassium nitrate}}{2KNO_3} + \underset{\text{sulphuric acid}}{H_2SO_4} = \underset{\text{potassium sulphate}}{K_2SO_4} + \underset{\text{nitric acid}}{2HNO_3}$$

(95) How much lead may be obtained from 932 kilograms of lead sulphide? The atomic weight of lead is 207 and the molecular weight of lead sulphide is 239. Ans. 807.21 Kg.

INDEX.

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